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Non-myopic learning in differential information economies: the core*

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Summary. We study the process of learning in a differential information economy, with a continuum of states of nature that follow a Markov process. The economy extends over an infinite number of periods and we assume that the agents behave non-myopically, i.e., they discount the future. We adopt a new equilibrium concept, the non-myopic core. A realized agreement in each period generates information that changes the underlying structure in the economy. The results we obtain serve as an extension to the results in Koutsougeras and Yannelis (1999) in a setting where agents behave non-myopically. In particular, we examine the following two questions: 1) If we have a sequence of allocations that are in an approximate non-myopic core (we allow for bounded rationality), is it possible to find a subsequence that converges to a non-myopic core allocation in a limit full information economy? 2) Given a non-myopic core allocation in a limit full information economy can we find a sequence of approximate non-myopic core allocations that converges to that allocation?

Keywords and Phrases: Non-myopic learning, Differential information, Core.

JEL Classification Numbers: D82, D50, D83.

1 Introduction

In this paper, we address the issue of learning in a differential information economy i.e., an economy with a finite number of agents, where each agent is characterized by a state dependent utility function, a state dependent initial endowment, a private information set (which is a partition of an exogeneously given probability measure space) and a prior. The equilibrium concept we employ is the

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non-myopic core which is an extension of the private core (see Yannelis, 1991) to dynamic economies with non-myopic agents. Our economy extends over an infinite number of periods and agents discount the future. Consequently, the utility functions depend not only on current consumption, but also on future consumption. Agents are risk averse and hence they want to smooth their consumption. In each period, they agree upon a contract that specifies the terms of exchange, contingent on the states of nature. This agreement is based on each agent's private information and has the property that there does not exist a coalition of agents who can redistribute their initial endowments using their private information and make everybody in the coalition better off. A realized agreement in each period generates information that changes the underlying information structure in the economy.

We are studying the exchange of goods and information that takes place at the interim stage i.e., after the agents have observed the events that contain the realized state of nature. To be more precise, all contracts are negotiated at the beginning of the history of the economy and from then on all actions are determined by the already chosen acts. There is no need to revise any strategies, because the choice of the strategies has already taken account of the structure of information in the future i.e., what information will be available at each date. The process through which learning occurs is the following: At the end of each period the agents observe the non-myopic core equilibrium outcome plus the endowments of the current period and they refine their information partitions. The link between today and the future is the information that each agent possesses. So, agents by deciding upon the trade that will take place today, affect their information partitions tomorrow, which in turn affects the future expected utility. Learning itself is not the goal of the agents, but rather a result of actions by agents who are concerned with the expected utility.

It becomes apparent from the above discussion that the information agents possess restricts their consumption and trade choices. A question that naturally arises, and the one we address is: Can the agents through the process of exchange reach a non-myopic core equilibrium allocation that is in a limit full information economy? (i.e., in an economy where everything that could be learned has been learned.)

This work draws upon the results obtained by Koutsougeras and Yannelis (1999). They addressed the issue of learning in a pure exchange economy with differential information using the private core (Yannelis, 1991) as an equilibrium concept by assuming that the agents behave myopically.[1] In their model agents only care about current consumption and their utility does not depend on future allocations at all.

There is a substantial literature that deals with the issue of non-myopic learning in dynamic games, a small subset of which are the papers by: Kalai and Lehrer (1993), Nyarko (1998) and Serfes and Yannelis (1998). However, they put the problem in a different setting than we do. In particular, the first two papers

[1] Henceforth, we will be calling the equilibrium concept in Koutsougeras and Yannelis, myopic core.

consider an infinitely repeated game where agents have subjective beliefs about their opponents' strategies. They prove convergence of the actual play to Nash equilibrium (Kalai and Lehrer), or convergence of beliefs to subjective Nash equilibria (Nyarko). Serfes and Yannelis (1998) address the same questions that are addressed in this paper in an infinitely repeated game setting by employing the Bayesian Nash as the equilibrium concept.

What we add to the existing literature and in particular to Koutsougeras and Yannelis (1999), is the study of the learning problem when agents behave non-myopically and the states of nature follow a Markov process. To do this, we introduce a new equilibrium concept, the non-myopic core. The result is that we may get allocations and learning processes that may differ, depending on the equilibrium concept i.e., myopic versus non-myopic core. Our equilibrium concept is more general, since as the discount factor goes to zero, our model reduces to that of Koutsougeras and Yannelis (1999).

The paper contains the following results: We prove the non-emptiness of the set of non-myopic core allocations. Next, we define the concept of a limit full information economy and ask the following: If we have a sequence of allocations that are in an approximate non-myopic core (allowing for bounded rationality), is it possible to find a subsequence that converges to a non-myopic core allocation in a limit full information economy? And given a non-myopic core allocation in a limit full information economy can we find a sequence of approximate non-myopic core allocations that converges to that allocation?

The rest of the paper is organized as follows. In Section 2, we have collected the results that we are going to use in the sequel. In Section 3, we present the myopic core. In Section 4, we outline our model and prove the existence theorem. In Section 5, we describe the learning process, present an example and state the main Theorems. Finally, in Section 6 we prove our main learning theorems.

2 Mathematical preliminaries

If X and Y are sets, the *graph* of the set-valued function (or correspondence), $\phi : X \to 2^Y$ is denoted by

$$G_\phi = \{(x, y) \in X \times Y : y \in \phi(x)\}.$$

Let $(\Omega, \mathscr{F}, \mu)$ be a complete, finite measure space, and X be a separable Banach space. The set-valued function $\phi : \Omega \to 2^X$ is said to have a *measurable graph* if $G_\phi \in \mathscr{F} \otimes \beta(X)$, where $\beta(X)$ denotes the Borel σ-algebra on X and $\otimes$ denotes the product σ-algebra. The set-valued function $\phi : \Omega \to 2^X$ is said to be *lower measurable* or just *measurable* if for every open subset V of X, the set

$$\{\omega \in \Omega : \phi(\omega) \cap V \neq \emptyset\}$$

is an element of $\mathscr{F}$.

Let $(\Omega, \mathscr{F}, \mu)$ be a finite measure space and X be a Banach space. Following Diestel-Uhl (1977) the function $f : \Omega \to X$ is called *simple* if there exist

$x_1, x_2, ..., x_n$ in X and $\alpha_1, \alpha_2, ..., \alpha_n$ in $\mathscr{F}$ such that $\sum_{i=1}^{n} x_i \chi_{\alpha_i}$ where $\chi_{\alpha_i}(\omega) = 1$ if $\omega \in \alpha_i$ and $\chi_{\alpha_i}(\omega) = 0$ if $\omega \notin \alpha_i$. A function $f : \Omega \to X$ is said to be μ-*measurable* if there exists a sequence of simple functions $f_n : \Omega \to X$ such that $\lim_{n\to\infty} \|f_n(\omega) - f(\omega)\| = 0$ for almost all $\omega \in \Omega$. A μ-*measurable* function $f : \Omega \to X$ is said to be *Bochner integrable* if there exists a sequence of simple functions $\{f_n : n = 1, 2, ...\}$ such that

$$\lim_{n\to\infty} \int_{\Omega} \|f_n(\omega) - f(\omega)\| d\mu(\omega) = 0.$$

In this case we define for each $E \in \mathscr{F}$ the integral to be

$$\int_E f(\omega)d\mu(\omega) = \lim_{n\to\infty} \int_E f_n(\omega)d\mu(\omega).$$

It can be shown (see Diestel-Uhl, 1977, Theorem 2, p.45) that if $f : \Omega \to X$ is a μ-measurable function then, f is Bochner integrable if and only if $\int_{\Omega} \|f(\omega)\| d\mu(\omega) < \infty$.

For $1 \le p < \infty$, we denote by $L_p(\mu, X)$ the space of equivalence classes of X-valued Bochner integrable functions $x : \Omega \to X$ normed by

$$\|x\|_p = (\int_{\Omega} \|x(\omega)\|^p d\mu(\omega))^{\frac{1}{p}}.$$

It is a standard result that normed by the functional $\|\cdot\|_p$ above, $L_p(\mu, X)$ becomes a Banach space (see Diestel-Uhl, 1977, p.50).

Let $X : \Omega \to 2^Y$, be a correspondence, where Y is a Banach space. Also let $u : \Omega \times Y \to R$ be a real-valued function. Ω can be decomposed into an atomless part Ω_1 and a countable union of atoms Ω_2. A result due to Balder and Yannelis (1993), [Theorem 2.8] says that if: 1. a.e. in Ω_1, $X(\omega)$ is convex and closed, 2. $u(\omega, \cdot)$ is concave and upper semicontinuous on $X(\omega)$, 3. $u(\omega, \cdot)$ is integrably bounded, 4. for all $\omega \in \Omega_2$, $X(\omega)$ is weakly closed, and 5. $u(\omega, \cdot)$ is weakly upper semicontinuous on $X(\omega)$ then,

$$U(x) = \int_{\Omega} u(\omega, x(\omega))d\mu(\omega)$$

is weakly upper semicontinuous on the weakly closed set $L_X = \{y \in L_1(\mu, Y) : y(\omega) \in X(\omega)$ and y is $\mathscr{F}$ − measurable$\}$.

Another result due to Balder and Yannelis (1993), [Theorem 2.1] tells us that if $X(\omega)$ is convex and closed a.e. in Ω_1 and weakly closed a.e. in Ω_2, then L_X is weakly closed.

Now we present some basic results on Banach lattices (see Aliprantis-Burkinshaw, 1985). Recall that a *Banach lattice* is a Banach space L equipped with an order relation $\ge$ (i.e., $\ge$ is reflexive, antisymmetric, and transitive relation) satisfying:

(i) $x \ge y$ implies $x + z \ge y + z$ for every z in L,
(ii) $x \ge y$ implies $\lambda x \ge \lambda y$ for all $\lambda \ge 0$,

(iii) for all x, y in L there exists a supremum (least upper bound) $x \vee y$ and an infimum (gretest lower bound) $x \wedge y$,
(iv) $|x| \geq |y|$ implies $\|x\| \geq \|y\|$ for all x, y in L.

As usual $x^+ = x \vee 0$, $x^- = (-x) \vee 0$ and $|x| = x \vee (-x) = x^+ + x^-$; we call x^+, x^- the *positive* and *negative* parts of x, respectively and $|x|$ the *absolute value* of x. The symbol $\|\cdot\|$ denotes the norm on L. If x, y are elements of the Banach lattice L, then we define the order interval $[x, y]$ as follows:

$$[x, y] = \{z \in L : x \leq z \leq y\}.$$

Note that $[x, y]$ is norm closed and convex (hence weakly closed). A Banach lattice L is said to have an *order continuous norm* if, $x_\alpha \downarrow 0$ [2] in L implies $\|x_\alpha\| \downarrow 0$. A very useful result that will play an important role in the sequel is that if L is a Banach lattice then the fact that L has order continuous norm is equivalent to weak compactness of the order interval $[x, y] = \{z \in L : x \leq z \leq y\}$ for every x, y in L [see for instance Aliprantis-Brown-Burkinshaw (1990), Theorem 2.3.8, p.104 or Lindenstrauss-Tzafriri (1979), p.28].

We note that Cartwright (1974) has shown that if X is a Banach lattice with order continuous norm (or equivalently X has weakly compact order intervals) then $L_1(\mu, X)$, has weakly compact order intervals, as well.

We close this section by defining the notion of a martingale and stating the martingale convergence theorem. Let I be a directed set and let $\{\mathscr{F}_i : i \in I\}$ be a monotone increasing net of sub σ-fields of $\mathscr{F}$ (i.e., $\mathscr{F}_{i_1} \subseteq \mathscr{F}_{i_2}$ for $i_1 \leq i_2, i_1, i_2$ in I). A net $\{x_i : i \in I\}$ in $L_1(\mu, X)$ is *a martingale* if

$$E(x_i | \mathscr{F}_{i_1}) = x_{i_1}, \forall i \geq i_1.$$

We will denote the above martingale by $\{x_i, \mathscr{F}_i\}_{i \in I}$. The proof of the following *martingale convergence theorem* can be found in Diestel-Uhl (1977, p.126). A martingale $\{x_i, \mathscr{F}_i\}_{i \in I}$ in $L_1(\mu, X)$ converges in the $L_1(\mu, X)$-norm if and only if there exists x in $L_1(\mu, X)$ such that $E(x | \mathscr{F}_i) = x_i$ for all $i \in I$. Finally, recall (see for instance Diestel–Uhl, 1977, p.129) that if the martingale $\{x_i, \mathscr{F}_i\}_{i \in I}$ converges in the $L_1(\mu, X)$-norm to $x \in L_1(\mu, X)$, it also converges almost everywhere, i.e., $\lim_{i \to \infty} x_i = x$ almost everywhere.

3 The Yannelis core

The definition of the core of an exchange economy with differential information is given as follows (see also Yannelis, 1991).

Let Y, which denotes the commodity space,[3] be a separable Banach lattice with an order continuous norm and Y_+ be its positive cone. Let $(\Omega, \mathscr{F}, \mu)$ be a probability space. An exchange economy with differential information,

[2] $x_\alpha \downarrow 0$, means that x_α is a decreasing net with $\inf_\alpha x_\alpha = 0$.

[3] It is important to note that even if we assume that our commodity space is $Y = R^l$, the space $L_p(\mu, R^l)$, $1 \leq p \leq \infty$ is still infinite dimensional (in view of the continuum of states). Hence, even with one good we still need to work with an infinite dimensional space.

$$\mathscr{E} = \{(X_i, u_i, e_i, \mathscr{F}_i, q_i) : i = 1, 2, ..., n\}$$

is a set of quintuples $(X_i, u_i, e_i, \mathscr{F}_i, q_i)$ where,
(1) $X_i : \Omega \to 2^{Y_+}$ is the *random consumption set* of agent i.
(2) $u_i : \Omega \times Y_+ \to R$ is the *random utility function* of agent i.
(3) $\mathscr{F}_i$ is a sub-σ-algebra of $(\Omega, \mathscr{F})$ which denotes the private information of agent i.
(4) $e_i : \Omega \to Y_+$ is the *random initial endowment* of agent i, $e_i(\cdot)$ is $\mathscr{F}_i$-measurable, Bochner integrable and $e_i(\omega) \in X_i(\omega)$ for all i, $\mu - a.e.$.
(5) $q_i : \Omega \to R_{++}$ is the *prior* of agent i, (i.e., q_i is the Radon-Nikodym derivative having the property that $\int_{t\in\Omega} q_i(t)d\mu(t) = 1$).

Denote by L_{X_i}, the *set of all Bochner integrable and $\mathscr{F}_i$-measurable selections from the consumption set* X_i of agent i, i.e.,

$$\begin{aligned} L_{X_i} = \; & \{x_i \in L_1(\mu, Y_+) : x_i : \Omega \to Y_+ \text{is } \mathscr{F}_i - \text{measurable} \\ & \text{and } x_i(\omega) \in X_i(\omega), \mu - a.e.\}. \end{aligned}$$

For each i, $(i = 1, 2, \ldots, n)$ denote by $E_i(\omega)$ the event in $\mathscr{F}_i$ containing the realized state of nature $\omega \in \Omega$ and suppose that $\int_{t\in E_i(\omega)} q_i(t)d\mu(t) > 0$. Given $E_i(\omega)$ in $\mathscr{F}_i$ define the *interim expected utility* of agent i, $V_i : \Omega \times L_{X_i} \to R$ by,

$$V_i(\omega, x_i) = \int_{k\in E_i(\omega)} u_i(k, x_i(k))q_i(k|E_i(\omega))d\mu(k)$$

where

$$q_i(k|E_i(\omega)) = \begin{cases} 0 & \text{if } k \notin E_i(\omega) \\ \frac{q_i(k)}{\int_{k\in E_i(\omega)} q_i(k)d\mu(k)} & \text{if } k \in E_i(\omega). \end{cases}$$

Below we give the definitions of the core and the ϵ-core of the above economy.

Definition 3.1 (Yannelis, 1991). We say that $x = (x_1, x_2, ..., x_n) \in \Pi_{i=1}^n L_{X_i}$ is a core allocation for $\mathscr{E}$ if,

i) $\sum_{i=1}^n x_i = \sum_{i=1}^n e_i$ and,

ii) it is not true that there exists $S \subset \{1, 2, ..., n\}$ and $y \in \Pi_{i\in S} L_{X_i}$, such that $\sum_{i\in S} y_i = \sum_{i\in S} e_i$, and $V_i(\omega, y_i) > V_i(\omega, x_i), \forall i \in S$ and for almost all ω.

Definition 3.2 (Yannelis, 1991). An allocation $x \in \Pi_{i=1}^n L_{X_i}$, is said to be an ϵ**-core allocation** for $\mathscr{E}$ if in addition to i) above it satisfies

ii') it is not true that there exists $S \subset \{1, 2, ..., n\}$ and $y \in \Pi_{i\in S} L_{X_i}$ such that $\sum_{i\in S} y_i = \sum_{i\in S} e_i$, and $V_i(\omega, y_i) > V_i(\omega, x_i) + \epsilon, \forall i \in S$ and for almost all ω.

Theorem 3.1 (Yannelis, 1991). *Suppose that an exchange economy with differential information satisfies for each agent i the following assumptions,*
(a.3.1) $X_i : \Omega \to 2^{Y_+}$ *is a convex, closed, non-empty valued and* $\mathscr{F}$*-measurable correspondence.*
(a.3.2) for each $\omega \in \Omega$, $u_i(\omega, \cdot)$ *is continuous and integrably bounded and,*
(a.3.3) for each $\omega \in \Omega$, $u_i(\omega, \cdot)$ *is concave.*

Then a private core allocation exists in $\mathscr{E}$.

4 The non-myopic core

Let T be a countable set denoting the time horizon. Let Y be a Banach lattice with an order continuous norm and $(\Omega, \mathscr{F})$ be a measurable space with initial probability measure λ_0 and *transition function* Q.[4] The set Ω contains the states of nature which follow a *Markov* process overtime. Let $(\Omega^\infty, \mathscr{F}^\infty, \mu^\infty(\lambda_0, \cdot))$ be an infinite product probability measure space. The interpretation is that any sequence of *shocks* will lie in this space and μ^∞ gives the probability of that sequence occurring. Each state in Ω^∞ determines the entire history of all aspects of the economy that are beyond the control of any of the agents (see Savage, 1974, Ch. 2, for a detailed discussion of this concept).

Now let $x_{it} : \Omega^\infty \to Y_+$ be a vector-valued function that denotes the allocation of agent i in period t contingent on the history of realizations up to that period. We denote by $\overline{x_i} = (x_{i1}, \ldots, x_{it}, \ldots)$ an infinite sequence of such vector-valued functions for agent i. By $\overline{x}$ we denote such a sequence for all agents. Hence, $\overline{x}$ can be viewed as a stochastic process on $(\Omega^\infty, \mathscr{F}^\infty, \mu^\infty(\lambda_0, \cdot))$. Also the endowments $e_{it} : \Omega^\infty \to Y_+, t = 1, 2, \ldots$ define a stochastic process on the same space. All contracts are negotiated at the beginning of the history of the economy, and from then on all actions are determined by already chosen strategies. Such strategies may, of course, take account of new information as it becomes available. An exchange economy with differential information is actually a sequence of economies

$$\{\mathscr{E}^t : t \in T\}$$

where for each t,

$$\mathscr{E}^t = \{(X_i, u_i, e_{it}, \mathscr{F}_{it}, q_i) : i = 1, 2, \ldots, n\}$$

is a set of quintuples $(X_i, u_i, e_{it}, \mathscr{F}_{it}, q_i)$ where,
(1) $X_i : \Omega^\infty \to 2^{Y_+}$, is a *random consumption correspondence of agent* i.
(2) $u_i : \Omega^\infty \times Y_+ \to R$, is a *state dependent utility function* of agent i.
(3) $\mathscr{F}_{it}$ is a sub-σ-algebra of $(\Omega^\infty, \mathscr{F}^\infty)$ which denotes the private information of agent i in period t.
(4) $e_{it} : \Omega^\infty \to Y_+$ is the *random initial endowment* of agent i in period t, $e_{it}(\cdot)$ is $\mathscr{F}_{it}$-measurable, Bochner integrable and $e_{it}(\omega^\infty) \in X_i(\omega^\infty)$ for all i, $\mu^\infty - a.e.$.
(5) $q_i : \Omega^\infty \to R_{++}$ is the *prior* of agent i, (i.e., q_i is the Radon–Nikodym derivative having the property that $\int_{k \in \Omega^\infty} q_i(k) d\mu^\infty(k) = 1$).

As in R.J. Aumann (1987), we assume that the economy is common knowledge.

Denote by $L_{X_{it}}$, the set of all *Bochner integrable and $\mathscr{F}_{it}$-measurable selections* from the consumption set X_i of agent i, in period t i.e.,

$$L_{X_{it}} = \{x_{it} \in L_1(\mu^\infty, Y_+) : x_{it} : \Omega^\infty \to Y_+ \text{ is } \mathscr{F}_{it} - \text{measurable and}$$

[4] A transition function is a function $Q : \Omega \times \mathscr{F} \to [0, 1]$ such that,
a. for each $\omega \in \Omega, Q(\omega, \cdot)$ is a probability measure on $(\Omega, \mathscr{F})$; and
b. for each $A \in \mathscr{F}, Q(\cdot, A)$ is a $\mathscr{F}$-measurable function.

$$x_{it}(\omega^{\infty}) \in X_i(\omega^{\infty}), \mu^{\infty} - a.e.\}.$$

Thus, $\bar{x_i} = (x_{i1}, \ldots, x_{it}, \ldots)$ is an element of $L_{\bar{X_i}} = L_{X_{i1}} \times \ldots \times L_{X_{it}} \times \ldots$.

For each i, $(i = 1, 2, \ldots, n)$ and $t \in T$, denote by $E_{it}(\omega^{\infty})$ the event in $\mathscr{F}_{it}$ containing the realized state of nature $\omega^{\infty} \in \Omega^{\infty}$ and suppose that $\int_{k \in E_{it}(\omega^{\infty})} q_i(k) d\mu(k) > 0$ for all $t \in T$.

For each i, $(i = 1, 2, \ldots, n)$ and ω^{∞} define the *total discounted interim expected utility* of agent i, $\bar{V_i} : \Omega^{\infty} \times L_{\bar{X_i}} \to R$ by

$$\bar{V_i}(\omega^{\infty}, \bar{x_i}) = \sum_{t=0}^{\infty} \delta^t \int_{k \in E_{it}(\omega^{\infty})} u_i(k, x_{it}(k)) q_i(k | E_{it}(\omega^{\infty})) d\mu(k) \qquad (4.1)$$

where $\delta \in [0, 1)$ is the discount factor and $q_i(k|E_{it}(\omega^{\infty}))$ was defined in Section 3.

We are now ready to define the first central notion of the paper.

Definition 4.1. We say that $\bar{x} = (\bar{x_1}, \bar{x_2}, \ldots, \bar{x_n}) \in \Pi_{i=1}^n L_{\bar{X_i}}$ is a *non-myopic core allocation* for the economy $\{\mathscr{E}^t : t \in T\}$ if,

(i) $\sum_{i=1}^n x_{it} = \sum_{i=1}^n e_{it}$, for all $t \in T$ and,

(ii) it is not true that there exist $S \subset \{1, 2, ..., n\}$ and $(\bar{y_i})_{i \in S} \in \Pi_{i \in S} L_{\bar{X_i}}$ such that $\sum_{i \in S} y_{it} = \sum_{i \in S} e_{it}$, for all $t \in T$ and $\bar{V_i}(\omega^{\infty}, \bar{y_i}) > \bar{V_i}(\omega^{\infty}, \bar{x_i}), \forall i \in S$ and for almost all ω^{∞}.[5]

Definition 4.2. We say that $\bar{x} = (\bar{x_1}, \bar{x_2}, \ldots, \bar{x_n}) \in \Pi_{i=1}^n L_{\bar{X_i}}$ is an *approximate or ϵ-non-myopic core allocation* for the economy $\{\mathscr{E}^t : t \in T\}$ if in addition to (i) above it satisfies,

(ii') it is not true that there exist $S \subset \{1, 2, ..., n\}$ and $(\bar{y_i})_{i \in S} \in \Pi_{i \in S} L_{\bar{X_i}}$ such that $\sum_{i \in S} y_{it} = \sum_{i \in S} e_{it}$, for all $t \in T$ and $\bar{V_i}(\omega^{\infty}, \bar{y_i}) > \bar{V_i}(\omega^{\infty}, \bar{x_i}) + \epsilon, \forall i \in S$ and for almost all ω^{∞}.

We are now ready to state our first main result:

Theorem 4.1. *Let $\{\mathscr{E}^t : t \in T\}$ be an exchange economy with differential information as defined above which satisfies the following assumptions, for each i, $(i = 1, 2, \ldots, n)$,*

(a.4.1) $X_i : \Omega^{\infty} \to 2^{Y_+}$ is convex, closed, non-empty valued, and $\mathscr{F}^{\infty}$-measurable correspondence.

(a.4.2) for each ω^{∞}, u_i is upper semicontinuous on $X_i(\omega^{\infty})$ and integrably bounded.

(a.4.3) for each ω^{∞}, u_i is concave.

Then the set of non-myopic core allocations for $\{\mathscr{E}^t : t \in T\}$ is a non-empty subset of $\Pi_{i=1}^n L_{\bar{X_i}}$.

Lemma 4.1. *Under assumptions (a.4.1)–(a.4.3), the total discounted interim expected utility $\bar{V_i}$ (4.1) is weakly upper semicontinuous for each i and for each ω^{∞}.*

Proof. By assumption, the utility function u_i is upper semicontinuous. Then, by Theorem 2.8 in Balder and Yannelis (1993) (see also Section 2),

[5] What we mean is "for μ^{∞} almost all $\omega^{\infty} \in \Omega^{\infty}$," but for convenience from now on we simply write "for almost all ω^{∞}."

$$\int_{k \in E_{it}(\omega^\infty)} u_i(k, x_{it}(k)) q_i(k|E_{it}(\omega^\infty)) d\mu(k)$$

is weakly upper semicontinuous.

By another application of the same Theorem and since T consists of a countable union of atoms, $\overline{V}_i$ is weakly upper semicontinuous as well. □

Next we present the proof of our Theorem.

Proof of Theorem 4.1. The set $L_{\bar{X}_i}$ is convex since each $L_{X_{it}}$ is convex. It is also weakly closed, since again each $L_{X_{it}}$ is weakly closed (see Thm. 2.1 in Balder and Yannelis (1993)). Now let's define an n-person game $\hat{V}$ by

$$\hat{V}(S) = \{x \in R^n : \text{ there exists an allocation } \bar{y} \in \mathscr{A}_S \text{ such that}$$
$$x_i \leq \overline{V}_i(\omega^\infty, \bar{y}_i), \forall i \in S \text{ and for almost all } \omega^\infty\}$$

where $\mathscr{A}_S$ is defined as

$$\mathscr{A}_S = \{\bar{y} \in \Pi_{i \in S} L_{\bar{X}_i} : \sum_{i \in S} y_{it} = \sum_{i \in S} e_{it}, \text{ for all } t \in T\}.$$

Notice that $\mathscr{A}_S$ is weakly compact because $\bar{y}_i \in [0, \sum e_{i1}] \times [0, \sum e_{i2}] \times \ldots, \forall i \in S$, the order intervals $[0, \sum_{i=1}^n e_{it}], \forall t \in T$ are weakly compact (by Cartwright's Theorem) and $\Pi_{i \in S} L_{\bar{X}_i}$ is weakly closed. It is also nonempty since $e_{it} \in L_{X_{it}}$, for all $t \in T$.

The n-person game satisfies the properties of Scarf's Theorem (Scarf, 1967). Notice that the comprehensiveness follows immediately. The fact that $\hat{V}$ is bounded from above follows from the fact that $\forall i \in N$, $\overline{V}_i$ is a weakly upper semicontinuous real-valued function (Lemma 4.1) on the non-empty, weakly compact set A_S.

We need to show that $\hat{V}(S)$ is closed. To this end, let a sequence $(x_1^k, ..., x_n^k)$ of some $\hat{V}(S)$ satisfy $(x_1^k, ..., x_n^k) \to (x_1, .., x_n)$ in R^n. We must show that $(x_1, .., x_n)$ belongs to $\hat{V}(S)$. For each k pick an allocation $(\bar{y}_1^k, \ldots, \bar{y}_n^k)$ satisfying $x_i^k \leq \overline{V}_i(\omega^\infty, \bar{y}_i^k), \forall i \in S$ and for almost all ω^∞, and $\sum_{i \in S} y_{it}^k = \sum_{i \in S} e_{it}$, for all $t \in T$. Since $y_{it}^k \in [0, e_t]$ (where $e_t = \sum_{i=1}^n e_{it}$, for all $t \in T$) holds for all i and all k and $[0, e_t]$ is weakly compact, we can assume by passing to an appropriate subsequence that $\bar{y}_i^k \to y_i$ weakly for all i. Clearly, $(y_1, ..., y_n)$ is an allocation and $\sum_{i \in S} y_{it} = \sum_{i \in S} e_{it}$, for all $t \in T$. Since $\overline{V}_i$ is weakly upper semicontinuous it follows that

$$x_i = \limsup_k x_i^k \leq \limsup_k \overline{V}_i(\omega^\infty, \bar{y}_i^k) \leq \overline{V}_i(\omega^\infty, y_i)$$

for all $i \in S$ and for almost all ω^∞. Therefore, $(x_1, ..., x_n) \in \hat{V}(S)$ and so each $\hat{V}(S)$ is closed. Hence the market game $(\hat{V}, N)$ is balanced and has therefore a non-empty core (Scarf's Theorem). Standard arguments now can be applied (see for instance Aliprantis, Brown and Burkinshaw, 1990, pp.48–49) to show that non-emptiness of the core of the game $(\hat{V}, N)$ implies non-emptiness of the core of the economy $\{\mathscr{E}^t : t \in T\}$. □

Also an *approximate or ϵ-private non-myopic core allocation* exists since the set of all non-myopic core allocations, denoted by $C(\{\mathscr{E}^t : t \in T\})$, is a subset of the set of all ϵ-private non-myopic core allocations denoted by $C_\epsilon(\{\mathscr{E}^t : t \in T\})$.

Next we turn to the question of learning.

5 Convergence and approximation theorems for the non-myopic private core and ϵ-non-myopic private core

5.1 The process of learning

Let T be any countably infinite set denoting the time horizon. We are going to study the learning process described by Koutsougeras and Yannelis (1999), by using the non-myopic core as the equilibrium concept of our economy. There are two advantages of using the non-myopic core. First, it is a general concept and one can recover all the fundamental results of Koutsougeras and Yannelis (1999) by simply letting the discount factor go to zero. Second, and most important, the agents in our framework look into the future which may capture allocations and learning processes that cannot be captured by the myopic core. We also make a further generalization by allowing the states of nature to follow a Markov process.

The economy extends over an infinite number of periods. Since the agents are risk-averse, they want to smooth their consumption. Therefore, in each period they agree upon a contract which specifies the terms of the exchange contingent upon the realized state of nature.

Hence, each agent's private information in each period is generated by his/her endowment in current and all past periods, his/her utility function and the equilibrium allocations in previous periods i.e.,

$$\mathscr{F}_{it} = \sigma(\{e_{it'}, t' = 1, \dots, t\}, u_i, \{x_{t'}, t' = 1, \dots, t-1\}).$$

In this scenario, the private information of agent i in period $t+1$ will be $\mathscr{F}_{it}$ together with the information that the endowment, the utility function and the private core allocations generate i.e.,

$$\mathscr{F}_{it+1} = \mathscr{F}_{it} \vee \sigma(e_{it+1}, x_t).$$

Clearly, in period $t+2$ the private information set of agent i will be, $\mathscr{F}_{it+2} = \mathscr{F}_{it+1} \vee \sigma(e_{it+2}, x_{t+1})$ and so on. Consequently, for each agent i and each time period, we have that

$$\mathscr{F}_{it} \subseteq \mathscr{F}_{it+1} \subseteq \mathscr{F}_{it+2} \subseteq \dots \quad .$$

The above expression represents a learning process for agent i and it generates a sequence of differential information economies i.e., $\{\mathscr{E}^t : t \in T\}$.

Next we define a *limit full information economy,*

$$\mathscr{E}^\infty = \{(X_i, u_i, e_{i\infty}, \bar{\mathscr{F}_i}, q_i) : i = 1, 2, ..., n\}$$

to be the set of quintuples $(X_i, u_i, e_{i\infty}, \overline{\mathscr{F}_i}, q_i)$ where, $\overline{\mathscr{F}_i} = \vee_{t=0}^{\infty} \mathscr{F}_{it}$ is the pooled information of agent i over the entire time horizon, X_i, u_i, q_i have been defined previously and $e_{i\infty}$ denotes an endowment function in a limit full information economy which is $\overline{\mathscr{F}_i}$-measurable.

Denote by $C(\mathscr{E}^\infty)$ and $C_\epsilon(\mathscr{E}^\infty)$ the *set of all limit full information non-myopic core allocation and the limit full information non-myopic ϵ-core allocation* respectively for the economy $\mathscr{E}^\infty$.

Throughout our analysis we will assume that a private information economy $\{\mathscr{E}^t : t \in T\}$ as well as a limit full information economy $\mathscr{E}^\infty$, satisfy the assumptions (a.4.1), (a.4.2) and (a.4.3) and therefore, $C(\{\mathscr{E}^t : t \in T\}) \neq \emptyset$ and $C(\mathscr{E}^\infty) \neq \emptyset$. Since, $C(\{\mathscr{E}^t : t \in T\}) \subset C_\epsilon(\{\mathscr{E}^t : t \in T\})$ the latter set is non-empty as well.

It is apparent that the information structure of the economy largely determines the resulting allocation. The example we present next illustrates the above argument as well as how the learning takes place in our economy.

5.2 Example

Consider the following two person economy ($I = \{1, 2\}$) with two commodities i, j, ($X = R_+^2$) and four different states ($\Omega = \{a, b, c, d\}$). To simplify the example, the economy extends to only two periods t and $t + 1$. To be consistent with our notation in the previous section the state space in each period is $Z = \{\omega_1, \omega_2\}$ and $\Omega = Z \times Z$. Hence, $a = \omega_1\omega_1, b = \omega_1\omega_2, \ldots$. The idea as Debreu (1960) puts it is the following: Nature makes a choice (state) from a number of possibilities (states). These possibilities are states at time $t+1$ (in our example). Once a state is given, atmospheric conditions, technological knowledge, natural disasters, . . . are determined for the entire period under consideration. At time t economic agents have some information about the state at $t + 1$ which will occur. This knowledge in our economy comes from observing the endowments. Additional knowledge in each period is acquired by the allocation in the previous period. This information can be described by a partition of the set of states at $t + 1$ into sets called events at t.

Each state occurs with probability $\frac{1}{4}$. The random initial (period t) endowment and private information sets are given by

$$e_{1t} = ((10, 0), (10, 0), (10, 0), (10, 0)), \mathscr{F}_{1t} = \{\{a, b, c, d\}\}$$

$$e_{2t} = ((0, 10), (0, 10), (0, 0), (0, 0)), \mathscr{F}_{2t} = \{\{a, b\}, \{c, d\}\}.$$

Note that the initial endowment of each agent is measurable with respect to his/her partition. The utility function of both agents is given by

$$u(\omega, x) = \sqrt{x}_i + \sqrt{x}_j, \text{ for all } \omega.$$

The agents before they observe any event they will agree on the following contract. In period t there will be no trade due to measurability constraints. Note that the net trade must be measurable with respect to each agent's private information at period t. Since agent 1 has trivial information the net trade must be constant across all states. But agent 2 has nothing to give at states c and d. Thus, the net trade must be zero. This implies that the allocation in period t is

$$x_{1t} = ((10,0),(10,0),(10,0),(10,0))$$

$$x_{2t} = ((0,10),(0,10),(0,0),(0,0)).$$

The information that this allocation generates is

$$\sigma(x_t) = \{\{a,b\},\{c,d\}\}.$$

Hence, the agents in the second period $(t+1)$ will possess the following information (we assume that the endowments are the same)

$$\mathscr{F}_{1t+1} = \mathscr{F}_{1t} \vee \sigma(x_t) = \{\{a,b\},\{c,d\}\}$$

$$\mathscr{F}_{2t+1} = \mathscr{F}_{2t} \vee \sigma(x_t) = \{\{a,b\},\{c,d\}\}$$

and the allocation in that period will be

$$x_{1t+1} = ((5,5),(5,5),(10,0),(10,0))$$

$$x_{2t} = ((5,5),(5,5),(0,0),(0,0)).$$

Notice that the allocation is measurable with respect to each agent's information and that both agents became better off. Therefore, the agreed upon contract is

$$x = (x_t, x_{t+1})$$

as described above.

Below we state and prove the main theorems of this section.

5.3 Learning theorems

We assume that the sequence of endowments satisfies the following condition: There exists $\sum_{i\in S} e_{i\infty} \in L_1(\mu, Y)$ such that for all $S \subset N$

$$E[\sum_{i\in S} e_{i\infty} | \wedge_{i\in S} \mathscr{F}_{it}] = \sum_{i\in S} e_{it}, \forall t \in T.$$

Theorem 5.3.1. *Let $\{\mathscr{E}^t : t \in T\}$ be a sequence of private information economies satisfying the following assumption:*

$\forall S \subset N$, where N is the set of agents, $\{\sum_{i\in S} e_{it}, \wedge_{i\in S}\mathscr{F}_{it}\}_{t\in T}$ is a martingale.

If the sequence $\{x_t : t \in T\}$ belongs to $C_\epsilon(\{\mathscr{E}^t : t \in T\})$, then we can extract a subsequence $\{x_{t_m} : m = 1, 2, \ldots,\}$ from the sequence x_t which converges weakly to $x^* \in C(\mathscr{E}^\infty)$.

Theorem 5.3.2. *Let $\{\mathscr{E}^t : t \in T\}$ be a sequence of private information economies satisfying the following assumptions:*

(i) $\forall i, \{e_{it}, \mathscr{F}_{it}\}_{t\in T}$ is a martingale,

(ii) $\{\sum_{i=1}^{n} e_{it}, \wedge_{i=1}^{n} \mathscr{F}_{it}\}_{t \in T}$ *is a martingale.*

Let x^* *be a limit full information non-myopic core allocation for the economy* $\mathscr{E}^\infty$*, i.e.,* $x^* \in C(\mathscr{E}^\infty)$*. Then, there exists a* $t' \in T$ *big enough and a sequence of allocations* $\{x_t : t \in T\}$ *such that* $\{x_t\}_{t \geq t'} \in C_\epsilon(\{\mathscr{E}^t : t \geq t'\})$ *and* $\{x_t\}_{t \in T}$ *converges in the* L^1*-norm to* x^**.*

An immediate conclusion of Theorem 1 is the following result.

Corollary 5.3.1. *Let* $\{\mathscr{E}^t : t \in T\}$ *be a sequence of private information economies satisfying assumption the of Theorem 5.3.1. If the sequence* $\{x_t : t \in T\}$ *belongs to* $C(\{\mathscr{E}^t :, t \in T\})$*, then we can extract a subsequence from the sequence* x_t *which converges weakly to* $x^* \in C(\mathscr{E}^\infty)$*.*

Discussion: In both Theorems, the aggregate endowments in the economy obey a "stability" property given by the Martingale assumption. For a more detailed discussion about the assumptions see Koutsougeras and Yannelis (1999). Theorem 5.3.1 states that non-myopic and non-fully-rational agents can, by repetition, reach an equilibrium allocation in an economy that everything that could be learned has been learned. Theorem 5.3.2 states the converse. That is, given an equilibrium allocation in such an economy, non-myopic and non-fully-rational agents will find the way through trading to reach that allocation. This may be viewed as a kind of "stability" property of the non-myopic core.

Remark 1. When the discount factor δ goes to zero, the above two Theorems reduce to the ones in Koutsougeras and Yannelis (1999) i.e., Theorems 3.3.1 and 3.3.2.

Remark 2. For the above two Theorems we want the total discounted interim expected utility (4.1) to be weakly continuous and not just weakly upper semi-continuous. If in addition we assume that for all i, $\mathscr{F}_{it}$ is a partition, and the utility function u_i is weakly continuous, then $\overline{V}_i$ is weakly continuous [for more details see Yannelis (1991), Claim 4.1 and Balder and Yannelis (1993), Corollary 2.9].

6 Proofs of the theorems

6.1 Proof of Theorem 5.3.1

For each i, let $\overline{L}_{X_i}$ be the set of all Bochner integrable and $\overline{\mathscr{F}_i}$-measurable selections from the consumption correspondence X_i i.e.,

$$\overline{L}_{X_i} = \{x_{i\infty} \in L_1(\mu^\infty, Y_+) : x_{i\infty} : \Omega^\infty \to Y_+ \text{ is } \overline{\mathscr{F}_i} - \text{measurable and}$$
$$x_{i\infty}(\omega^\infty) \in X_i(\omega^\infty), \mu^\infty - a.e.\}.$$

An allocation in a limit full information economy belongs in the above set i.e., $x_i^* \in \overline{L}_{X_i}$. Let $e_{i\infty} \in \overline{L}_{X_i}$,[6] be the endowments for agent i in a limit full

[6] Notice, that since we are working with partitions, the allocations are essentially in l^1.

information economy. Note that for each t, each feasible consumption $x_{it} \in L_{X_{it}}$, lies in the order interval $[0, \sum_{i=1}^{n} e_{it}] \subset \sum_{i=1}^{n} L_{X_{it}}$. By the *Cartwright Theorem*, $[\underline{0}, e_t]$ (and any order interval) (where $e_t = \sum_{i=1}^{n} e_{it}$), is weakly compact. Finally, $\overline{V}_i(\omega^\infty, \cdot)$ is *weakly continuous* for each i and for each ω^∞.

Let $\overline{x} = \{x_t : t \in T\}$ be in $C_\epsilon(\{\mathscr{E}^t : t \in T\})$. Obviously, $x_{it} \in [0, \sum_{i=1}^{n} e_{it}]$, for all i and $t \in T$. Since e_t is a martingale, it converges to e_∞ in the $L_1(\mu, Y)$-norm. Moreover, $L_1(\mu, Y)$ is a Banach lattice. By a standard result [e.g. Aliprantis-Burkinshaw (1985)] we can extract a subsequence (for convenience we still denote it by e_t) and find a positive element z in $L_1(\mu, Y)$ such that $|e_t - e_\infty| < \frac{1}{2^k} z$, (where the superscript k is the index of the subsequence). Hence, we can conclude that the subsequence e_t is order bounded above by an element say v in $L_1(\mu, Y)$ and below by 0 i.e., e_t belongs to the order interval $[0, v]$ in $L_1(\mu, Y)$. Therefore, a subsequence of the allocation $\overline{x} = \{x_t : t \in T\}$ belongs to the order interval $[0, v]$. By the *Eberlein-Smulian Theorem* we can extract a further subsequence (still denoted by $\{x_t\}$) which converges weakly to $x^* \in [0, v]^n$ (where $[0, v]^n$ is the n-fold product of $[0, v]$). Notice that in what follows we are dealing with the subsequence of the subsequence of the original allocation $\overline{x}$ and endowments e_t for which we kept the same indices.

We need to show that x^* is in $C(\mathscr{E}^\infty)$. Note that for each $t \in T, \sum_{i=1}^{n} x_{it} = \sum_{i=1}^{n} e_{it}$, $\{x_t\}_{t \in T}$ converges weakly to x^* and $\{e_t\}_{t \in T}$ converges weakly to e_∞ (since by the Martingale Convergence Theorem e_{it} converges in the L^1-norm to $e_{i\infty}$ and hence weakly). So, we conclude that $\sum_{i=1}^{n} x_i^* = \sum_{i=1}^{n} e_{i\infty}$. Thus, $x_i^* \in [\underline{0}, e_\infty] \subset \sum_{i=1}^{n} \overline{L}_{X_i}$ (where $e_\infty = \sum_{i=1}^{n} e_{i\infty}$), and therefore each x_i^* is $\mathscr{F}_i$-measurable.

Hence, all it remains to be shown is that

there is no coalition S and $y_\infty \in \Pi_{i \in S} \overline{L}_{X_i}$ such that $\sum_{i \in S} y_{i\infty} = \sum_{i \in S} e_{i\infty}$, and $\overline{V}_i(\omega^\infty, y_{i\infty}) > \overline{V}_i(\omega^\infty, x_i^*), \forall i \in S$ and for almost all ω^∞.

Suppose by way of contradiction that this is not true. Then, there exists a coalition S and $y_\infty \in \Pi_{i \in S} \overline{L}_{X_i}$ such that $\sum_{i \in S} y_{i\infty} = \sum_{i \in S} e_{i\infty}$, and $\overline{V}_i(\omega^\infty, y_{i\infty}) > \overline{V}_i(\omega^\infty, x_i^*), \forall i \in S$ and for almost all ω^∞.

For each $i \in S$ and each $t \in T$, let $y_{it} = E[y_{i\infty} | \wedge_{i \in S} \mathscr{F}_{it}]$. Notice that

$$E[y_{i\infty} | \wedge_{i \in S} \mathscr{F}_{it}] = E[E[y_{i\infty} | \wedge_{i \in S} \mathscr{F}_{it'}] | \wedge_{i \in S} \mathscr{F}_{it}] = E[y_{it'} | \wedge_{i \in S} \mathscr{F}_{it}], \text{ for } t' \geq t.$$

Hence, $\{y_{it}, \wedge_{i \in S} \mathscr{F}_{it}\}_{t \in T}$ is a martingale and

$$\begin{aligned} \sum_{i \in S} y_{it} &= \sum_{i \in S} E[y_{i\infty} | \wedge_{i \in S} \mathscr{F}_{it}] = E\left[\sum_{i \in S} y_{i\infty} | \wedge_{i \in S} \mathscr{F}_{it}\right] \\ &= E\left[\sum_{i \in S} e_{i\infty} | \wedge_{i \in S} \mathscr{F}_{it}\right] = \sum_{i \in S} e_{it}. \end{aligned}$$

This is true for all $t \in T$ and hence $\{y_{it}\}_{t \in T}$ is feasible for the coalition S.

By virtue of the Martingale Convergence Theorem, $\{y_{it}\}_{t \in T}$ converges to $y_{i\infty}$ in the L_1-norm and therefore weakly. Since $\{x_t\}_{t \in T}$ also converges weakly to x^* and $\overline{V}_i$ is weakly continuous we may choose $t' \in T$ so that

$$|\bar{V}_i(\omega^\infty, y_{i\infty}) - \bar{V}_i(\omega^\infty, \{y_{it}\}_{t\geq t'})| < \frac{\delta-\epsilon}{2} \quad \text{and}$$
$$|\bar{V}_i(\omega^\infty, \{x_{it}\}_{t\geq t'}) - \bar{V}_i(\omega^\infty, x_i^*)| < \frac{\delta-\epsilon}{2}$$

where $\delta = \bar{V}_i(\omega^\infty, y_i^\infty) - \bar{V}_i(\omega^\infty, x_i^*) > \epsilon$. Thus,

$$\begin{aligned}&|\bar{V}_i(\omega^\infty, y_{i\infty}) - \bar{V}_i(\omega^\infty, \{y_{it}\}_{t\geq t'}) + \bar{V}_i(\omega^\infty, \{x_{it}\}_{t\geq t'}) - \bar{V}_i(\omega^\infty, x_i^*)|\\ &\leq |\bar{V}_i(\omega^\infty, y_{i\infty}) - \bar{V}_i(\omega^\infty, \{y_{it}\}_{t\geq t'})| + |\bar{V}_i(\omega^\infty, \{x_{it}\}_{t\geq t'}) - \bar{V}_i(\omega^\infty, x_i^*)|\\ &< \frac{\delta-\epsilon}{2} + \frac{\delta-\epsilon}{2} = \delta - \epsilon.\end{aligned}$$

Therefore, $\bar{V}_i(\omega^\infty, y_{i\infty}) - \bar{V}_i(\omega^\infty, \{y_{it}\}_{t\geq t'}) + \bar{V}_i(\omega^\infty, \{x_{it}\}_{t\geq t'}) - \bar{V}_i(\omega^\infty, x_i^*) < \delta - \epsilon \iff -\bar{V}_i(\omega^\infty, \{y_{it}\}_{t\geq t'}) + \bar{V}_i(\omega^\infty, \{x_{it}\}_{t\geq t'}) < -\epsilon$ or $\epsilon + \bar{V}_i(\omega^\infty, \{x_{it}\}_{t\geq t'}) < \bar{V}_i(\omega^\infty, \{y_{it}\}_{t\geq t'}), \forall i \in S$ and for almost all ω^∞.

So, the allocation y_t is feasible and $\bar{V}_i(\omega^\infty, \{y_{it}\}_{t\geq t'}) > \bar{V}_i(\omega^\infty, \{x_{it}\}_{t\geq t'}) + \epsilon, \forall i \in S$ and for almost all ω^∞, a contradiction to the fact that $\bar{x} \in C_\epsilon(\{\mathscr{E}^t : t \in T\})$. □

6.2 Proof of Theorem 5.3.2

Let x_∞ be an element of $C(\mathscr{E}^\infty)$. Consider the allocation $x_t = E[x_\infty | \wedge_{i=1}^n \mathscr{F}_{it}]$ and notice that for $r \geq t$

$$x_t = E[x_\infty | \wedge_{i=1}^n \mathscr{F}_{it}] = E[E[x_\infty | \wedge_{i=1}^n \mathscr{F}_{ir}] | \wedge_{i=1}^n \mathscr{F}_{it}] = E[x_r | \wedge_{i=1}^n \mathscr{F}_{it}].$$

Hence $\{x_t, \wedge_{i=1}^n \mathscr{F}_{it}\}_{t\in T}$ is a martingale and by virtue of the Martingale Convergence Theorem $\{x_t\}_{t\in T}$ converges in the L^1-norm to x_∞. By the definition of the conditional expectation we know that for each i and $t \in T$, x_{it} is $\mathscr{F}_{it}$-measurable. We must show that there exists a t' big enough such that the sequence $\{x_t : t \geq t'\}$ lies in $C_\epsilon(\{\mathscr{E}^t : t \geq t'\})$. We first show that $\{x_t\}_{t\in T}$ is feasible for the grand coalition. Note that, for all $t \in T$

$$\begin{aligned}\sum_{i=1}^n x_{it} &= \sum_{i=1}^n E[x_{i\infty} | \wedge_{i=1}^n \mathscr{F}_{it}] = E[\sum_{i=1}^n x_{i\infty} | \wedge_{i=1}^n \mathscr{F}_{it}] = E[\sum_{i=1}^n e_{i\infty} | \wedge_{i=1}^n \mathscr{F}_{it}]\\ &= \sum_{i=1}^n e_{it},\end{aligned}$$

and we can conclude that $\{x_t\}_{t\in T}$ is feasible. We now show that there exists a t' such that the allocation $\{x_t\}_{t\geq t'}$ cannot be ϵ-blocked by any coalition i.e.,

there do not exist coalition S and allocation $\{y_t\}_{t\geq t'} \in \Pi_{i\in S} L_{\bar{X}_i}$ such that $\sum_{i\in S} y_{it} = \sum_{i\in S} e_{it}$, for all $t \geq t'$ and $\bar{V}_i(\omega^\infty, \{y_{it}\}_{t\geq t'}) > \bar{V}_i(\omega^\infty, \{x_{it}\}_{t\geq t'}) + \epsilon, \forall i \in S$ and for almost all ω^∞.

Suppose by way of contradiction that the above statement is false. Then, there exists a coalition S and a sequence $\{y_t\}_{t\in T}$, $y_t \in \Pi_{i\in S} L_{X_{it}} \subset \Pi_{i\in S} L_{\bar{X}_i}$ having the property that $\sum_{i\in S} y_{it} = \sum_{i\in S} e_{it}$, for all $t \in T$ and $\bar{V}_i(\omega^\infty, \{y_{it}\}_{t\geq t'}) > \bar{V}_i(\omega^\infty, \{x_{it}\}_{t\geq t'}) + \epsilon, \forall i \in S$ for almost all ω^∞ and for all t'.

By adopting an argument similar to the one in the previous proof, $\{y_t\}_{t\in T}$ lies in the order interval $[0,v]^{|S|}$ which is weakly compact (recall Cartwright's Theorem). Hence, by the weak compactness of $[0,v]^{|S|}$ we can find a further subsequence $\{y_{t_m}\}$ that converges weakly to $y_\infty \in [0,v]^{|S|}$. For this subsequence we have

$$\sum_{i\in S} y_{it_m} = \sum_{i\in S} e_{it_m}, \text{ for all } t_m \in T.$$

Since e_{it_m} converges to $e_{i\infty}$ in the L^1-norm and hence weakly and y_{it_m} converges to $y_{i\infty}$ weakly we have that

$$\sum_{i\in S} y_{i\infty} = \sum_{i\in S} e_{i\infty}.$$

By our assumption, we also have that

$$\bar{V}_i(\omega^\infty, \{y_{it_m}\}_{t_m\geq t'}) > \bar{V}_i(\omega^\infty, \{x_{it_m}\}_{t_m\geq t'}) + \epsilon$$

$\forall i \in S$, for almost all ω^∞ and for all t'. By the weak continuity of $\bar{V}_i(\omega^\infty,\cdot)$, $\bar{V}_i(\omega^\infty, y_{i\infty}) \geq \bar{V}_i(\omega^\infty, x_{i\infty}) + \epsilon, \forall i \in S$ and for almost all ω^∞. Hence, $\bar{V}_i(\omega^\infty, y_{i\infty}) > \bar{V}_i(\omega^\infty, x_{i\infty}), \forall i \in S$ and for almost all ω^∞ and consequently the coalition S qualifies to block x_∞ a contradiction to the fact that $x_\infty \in C(\mathscr{E}^\infty)$.
□

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