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ORIGINAL PAPER

Endogenous matching in a market with heterogeneous principals and agents

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Abstract We employ the assignment game of Shapley and Shubik (Int J Game Theory 1:111–130, 1972) to study the endogenous matching patterns in a market that consists of heterogenous principals and agents. We show that, in general, the equilibrium matching is non-assortative. We then characterize the equilibrium relationship between risk and performance pay and risk and fixed compensation. This is the first paper that characterizes the equilibrium matching, to its fullest possible extent, building on the Holmstrom and Milgrom (Econometrica 55:303–328, 1987) principal-agent model. This model has been used extensively in the empirical literature and therefore we hope that our results will be of value to empirical researchers who wish to study a principal-agent market.

Keywords Risk sharing · Endogenous matching · Assignment Game

JEL Classification C78 · D81

1 Introduction

In a principal-agent model under moral hazard the optimal contract balances optimally risk sharing with incentives for effort. This generates a negative relationship between

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risk and incentives, e.g. Holmstrom and Milgrom (1987). If risk increases, the principal will offer a contract with less incentives for effort and higher insurance, and vice versa. This prediction has generated a voluminous empirical literature across many fields and markets which attempts to verify the validity of the risk sharing model. In particular, the risk sharing model has been applied extensively in areas such as, executive compensation, sharecropping and franchising [Prendergast (2002) offers an excellent survey of the empirical literature]. Few papers, however, have confirmed the negative relationship empirically, while the majority of them either discovered a positive or a non-significant relationship between risk and performance pay. Most of the earlier literature on this issue ignored the possibility that a principal-agent pair is an outcome of an endogenous matching in a market which consists of heterogeneous principals and agents. The empirical data, on the other hand, are generated from a principal-agent market, rather than an isolated principal-agent pair. More recently, a small number of papers have addressed the issue of endogenous matching, e.g. Ackerberg and Botticini (2002), Legros and Newman (2002), Wright (2004), Besley and Ghatak (2005), Serfes (2005) and Dam and Perez-Castrillo (2006).

The present paper extends the existing literature on endogenous matching in principal-agent markets. In particular, we use the assignment game (Shapley and Shubik 1972) to analyze the matching patterns in a market that consists of heterogeneous principals and agents. Serfes (2005) studied the relationship between risk and performance pay (incentives) in a principal-agent market. He found conditions under which the equilibrium relationship between risk and incentives is negative, positive, or non-monotonic. This paper extends Serfes (2005) in several directions. First, Serfes (2005) assumed a continuum of principals and agents with uniform distributions and did not use the assignment game. In the present paper, the number of principals and agents is finite. Hence, we can employ the assignment game, which in turn allows us to characterize the problem to its fullest possible extent. Second, Serfes (2005) found sufficient conditions for monotone and non-monotone matching, but did not go any further. In this paper we can use the assignment game and its algorithms to compute the equilibrium matching in any given principal-agent market. This is useful, especially for empirical researchers working on this issue. Third, Serfes (2005) was mainly interested in the relationship between risk and incentives, whereas in the present paper this is only one of the questions we examine. Other important issues that are addressed in this paper are: (i) the properties of the equilibrium matching and (ii) the relationship between risk and the fixed compensations.[1] Fourth, the production and cost functions in the present paper are more general than in Serfes (2005).

Dam and Perez-Castrillo (2006) also use the assignment game to analyze the endogenous matching in a principal-agent market. Our model differs from theirs in the following respects. First, in their model all players are risk neutral and the tension comes from a limited liability constraint, whereas in our model there is no limited liability and the agents are risk averse. In other words, our model is tailored to the Holmstrom and Milgrom (1987) risk sharing model, which has found a wide applicability. Second, we focus explicitly on the relationship between risk and incentives

[1] That relationship can be used by empirical researchers (together with the relationship between risk and incentives) to sort out competing theories.

(and risk and fixed compensation), while they study other issues. Wright (2004), using a model that is very similar to ours, studies the matching patterns in a principal-agent market. In his model there are two types of principals and agents, while in our model there are many types. Furthermore, the production and cost functions in Wright (2004) are not general, which is the case in our model. Finally, Wright (2004) does not use explicitly the assignment game and the algorithms associated with this game. Legros and Newman (2002) find sufficient conditions for monotone matching. Besley and Ghatak (2004) study a principal-agent matching model in the presence of motivated agents in organizations. In an empirical paper, Ackerberg and Botticini (2002) test: (i) the predictions of the risk sharing model and (ii) for endogenous matching. Using a historical dataset on agricultural contracts from Renaissance Tuscany, they find strong evidence for endogenous matching between landlords and tenants and that risk sharing is an important determinant of contract choice.

We build on the standard risk sharing model (e.g. Holmstrom and Milgrom 1987) by introducing a market with several principals and agents. Agents have diverse attitudes towards risk, while principals are risk neutral and each one's asset is subject to a different exogenous variability. Each principal forms a partnership with one agent by offering him a (linear) contract which specifies: (i) the share of the output that the agent keeps (incentives), (ii) a fixed salary and (iii) a fixed bonus offered to the agent on top of his reservation payoff.[2] Contracts in this setting have a dual role: first to balance incentives for effort with risk sharing optimally and second to facilitate an efficient matching between the principals and the agents. The second role of a contract has not received much attention in the literature, until recently. Contract theory has been primarily concerned with a contract design (in a single principal-agent, common agent, or multi-agent framework) which generates the highest surplus given certain informational asymmetries. An equally important issue, however, is how the competition in a principal-agent market shapes up the characteristics of equilibrium contracts and their profile across the participants. In our setting the competition is among principals who compete for low risk averse agents, which results in an outcome where *both* the agents and the principals receive rents, even when principals have all the bargaining power.[3] The equilibrium market clearing price (bonus) that a principal has to pay to hire an agent is inversely related to the agent's degree of risk aversion. A matching between the principals and the agents along with a menu of contracts constitutes an equilibrium if no individual principal or a principal-agent pair can profitably deviate by signing a new contract. Using the assignment game (Shapley and Shubik 1972), we show that such an equilibrium exists and we characterize it. We also show how an equilibrium matching can be computed using the linear programming algorithm of the assignment game.

The endogenous matching that takes place gives rise to an equilibrium principal-agent assignment, which is captured by the "matching curve" (i.e., equilibrium variance-risk aversion pairs). Higher productivity principals (agents) are the ones with lower variance assets (degrees of risk aversion). The matching is *positively assortative*

[2] When we say a fixed compensation we mean the sum of the fixed salary and the fixed bonus. The bonus does not depend on the realization of the uncertainty.

[3] The same is true in Dam and Perez-Castrillo (2006).

(PAM) (e.g. Becker 1973; Shimer and Smith 2000) if high productivity principals are matched with high productivity agents, i.e., the matching curve has a positive slope. It is *negatively assortative (NAM)* if low productivity principals are matched with high productivity agents, i.e., the matching curve has a negative slope. We find conditions under which a matching is: (i) positively assortative, (ii) negatively assortative, or (iii) both (*not globally assortative, NGAM*), (Sect. 4.2).[4] As a result, higher risk does not necessarily imply lower incentives, which is the prediction in Holmstrom and Milgrom (Sect. 5).[5]

We also characterize the properties of the equilibrium matching (Sect. 4.3). A minimum price competitive equilibrium (MPCE) is an equilibrium where the agents' fixed compensations are the lowest possible. Interestingly, when the equilibrium matching is assortative (either PAM or NAM), then the MPCE is the unique outcome. Therefore, the side of the market that has all the bargaining power (i.e., the principals) will obtain the most favorable outcome. Under the assumption of an assortative matching, we present a simple way of calculating the fixed compensations. If the matching is not globally assortative (NGAM), then the MPCE is *not* unique. This suggests that, under a NGAM, the MPCE is *not* guaranteed to the side of the market that has all the bargaining power. Moreover, in this case, we present a general linear programming algorithm, taken from Leonard (1983), that computes the MPCE.

This paper contributes to the contract design and the endogenous matching literature. By incorporating the analysis of an isolated principal-agent pair into the principal-agent market, we discover interesting results which advance our knowledge on how these markets function. Our theoretical framework can be used as a guide for empirical models which test how principals are matched with agents and how this matching affects the provision of incentives and the fixed compensations. Moreover, the matching literature has mainly followed two approaches. One was pioneered by Shapley and Shubik (1972) where the participants' preferences are very general and the other was introduced by Becker (1973) where strong assumptions are made about the nature of the preferences (e.g. either a supermodular or a submodular production function, which leads to a positively or a negatively assortative matching, respectively). The present paper profitably combines these two approaches in the specific "risk sharing" framework.[6] On the one hand, our model is structural, but on the other hand the preferences are not taken as primitives, but rather derived from the data of our problem.

Our paper is the first paper that characterizes the equilibrium matching, to its fullest possible extent, building on the Holmstrom and Milgrom (1987) principal-agent model. This model has been used extensively in the empirical literature and therefore we hope that our results will be of value to empirical researchers.

The rest of the paper is organized as follows. Section 2 presents the model. Section 3 characterizes the equilibrium properties when an isolated principal-agent pair

[4] Anderson and Smith (2003) show that non-assortative matching can arise in a dynamic model where the output is stochastic and the matching is between agents with unobserved productivity, even in the presence of production complementarities.

[5] The intuition for this result can be found in Serfes (2005, p. 344).

[6] See Roth and Sotomayor (1990, p. 247, para 8), for a short discussion on this point.

is analyzed. Section 4 analyzes a general market with n principals and n agents and presents the main results. Based on the results in Sects. 4, and 5 studies the equilibrium relationship between risk and incentives and risk and fixed compensations. Section 6 offers three concrete examples which demonstrate how one can compute the equilibrium matching and the terms of the contracts. These examples should be useful to empirical researchers. We end in Sect. 7 with a few concluding remarks. Most of the proofs are in the appendix.

2 The description of the model

The standard risk model in agency theory (e.g. Holmstrom and Milgrom 1987) is extended in only one direction, that is, to introduce a population of heterogeneous principals and agents. In particular, we consider a market which consists of P principals indexed by p ($p \in P = \{1, \dots, P\}$) and A agents indexed by a ($a \in A = \{1, \dots, A\}$).[7] We assume that there are more agents than principals, i.e., $P \leq A$ and that each principal p has to hire an agent a to manage his asset (e.g. firm, land, outlet, project etc.).[8]

Definition 1 (*Matching; Roth and Sotomayor 1990*) A matching m is a one-to-one correspondence from the set $P \cup A$ onto itself of order two,[9] such that if $m(p) \neq p$ then $m(p) \in A$ and if $m(a) \neq a$ then $m(a) \in P$.

This definition allows for the possibility that a principal (or an agent) may remain unmatched, in which case he is matched with himself and receives his reservation utility. We say that principal p is matched with agent a if $m(p) = a$.

Denote a given pair by (p, a). The production function for this specific pair is $y_{p,a} = f(e_a) + \varepsilon_p$, where, (i) e_a denotes the effort that the agent exerts, (ii) ε_p is a random shock which is distributed normally and independently of $\varepsilon_{p'}$, $p' \in P$, with mean $\mu_p \geq 0$ and variance σ_p^2, and (iii) $y_{p,a}$ is the output produced. We assume that the per-unit output price is the same across all principal-agent pairs and we furthermore normalize it to one. We assume that $f(\cdot)$ is continuously differentiable with $f(0) = 0$, $f^{'}(0) > 0$, $f^{'} > 0$, and $f^{''} \leq 0$. Effort is costly, with its cost being $C(e_a)$, where $C(\cdot)$ is continuously differentiable with $C(0) = 0$, $C^{'}(0) = 0$, $C^{'} > 0$, and $C^{''} > 0$. The wage of agent a employed by principal p is denoted $w_{p,a}$. Agents are risk averse, while the principals are risk neutral. Agent a's utility function is given by $V = 1 - \exp[-r_a(w_{p,a} - C(e_a))]$, where $r_a > 0$ is his degree of absolute risk aversion, which varies across agents. Let $\boldsymbol{\sigma}^2 = (\sigma_1^2, \dots, \sigma_P^2)$ and $\mathbf{r} = (r_1, \dots, r_A)$ denote the vectors of the principals' asset variances and the agents' degrees of risk aversion, respectively. Moreover, we assume that $0 < \sigma_1^2 < \dots < \sigma_P^2 < \infty$ and $0 < r_1 < \dots < r_A < \infty$.

[7] Gretsky et al. (1992) solve the assignment game with a continuum of players. Although a continuum of players may simplify the analysis in some respects, it will introduce complications in some other dimensions. For example, with a finite number of players, as we show in subsequent sections, the optimal assignment can be derived easily, using a simple algorithm which can be found in most software packages, e.g., Maple.

[8] It seems reasonable to assume that the potential number of assetless agents is no less than the number of principals who own assets.

[9] "Order two" means that if principal p is matched to an agent a, then agent a is matched to principal p.

In this setting, Holmstrom and Milgrom (1987) showed that the optimal compensation scheme—for a fixed pair (p, a)—is linear and is given by $w_{p,a} = B_{p,a} + \alpha_{p,a} y_{p,a}$, where $B_{p,a}$ is agent a's fixed compensation and $\alpha_{p,a}$ $(0 \leq \alpha_{p,a} \leq 1)$ is his share of output. We further assume that each agent's and each principal's reservation utility is zero and that the principals have all the bargaining power. Principal p's expected profit when he employees agent a is, $\bar{\Pi}_{p,a} = (1 - \alpha_{p,a}) \left[f(e_a) + \mu_p \right] - B_{p,a}$.

It will facilitate the subsequent analysis if we decompose the fixed compensation $B_{p,a}$ into two parts, $\beta_{p,a}$ and $b_{p,a}$, such that $B_{p,a} = \beta_{p,a} + b_{p,a}$. The first part, $\beta_{p,a}$ is the standard salary which is chosen by the principal to satisfy the agent's binding participation constraint. The second part, $b_{p,a}$, is a bonus (with some abuse of the term) which is a fixed monetary transfer, in excess of the agent's reservation utility, from principal p to agent a. The role of the bonus in our setting is to clear the market, i.e., to equalize the demand for agents with a given degree of risk aversion with the supply (more on this in Sect. 4). We assume that unmatched players sign a null contract and receive their reservation payoffs. Given a matching m a contract signed between principal p and agent a is denoted by $\mathcal{C}_{p,a} = \langle \alpha_{p,a}, B_{p,a} \rangle$. Let $\mathcal{C}$ denote the menu of contracts induced by a matching m. Finally, we call the pair $(\mathcal{C}, m)$ a market outcome.

The game we consider unfolds as follows:

Stage 1. Each principal $p \in P$ is being matched with exactly one agent $a \in A$, who is offered the contract $\mathcal{C}_{p,a}$.

Stage 2. Given $(\mathcal{C}, m)$ each agent chooses a level of effort (which is unobserved by the principal).

The random variable ε_p is realized after all the decisions have been made.

Consider a market outcome $(\mathcal{C}, m)$ where there exists a principal p who is currently matched with agent a, i.e., $m(p) = a$. Let $u_p(\mathcal{C}_{p,a}) = \bar{\Pi}_{p,a}$ and $v_a(\mathcal{C}_{p,a}) = 1 - E\left[\exp\left\{-r_a\left(w_{p,a} - C(e_a)\right)\right\}\right]$, be principal p's and agent a's expected utilities, respectively. We will consider two different types of blockings:

Definition 2 (*Pair blocking*) A market outcome $(\mathcal{C}, m)$ is blocked by a pair, if there are two principal-agent pairs, $m(p) = a$ and $m(p') = a'$, such that both principal p and agent a' will become better off if they break their current assignment and match with each other under a new contract $\mathcal{C}'_{p,a'}$. That is, $u_p(\mathcal{C}'_{p,a'}) > u_p(\mathcal{C}_{p,a})$ and $v_{a'}(\mathcal{C}'_{p,a'}) > v_{a'}(\mathcal{C}_{p',a'})$.

Definition 3 (*Principal blocking*) A market outcome $(\mathcal{C}, m)$ is blocked by a principal p (where $m(p) = a$), if he offers a new contract $(\mathcal{C}'_{p,a})$ to his assigned agent a and becomes better off, i.e., $u_p(\mathcal{C}'_{p,a}) > u_p(\mathcal{C}_{p,a})$. Agent a may not necessarily become weakly better off.

The first blocking refers to a situation where a principal becomes strictly better off by hiring a different agent (who also becomes strictly better off) and offering him a new contract. An immunity from this type of blocking is the standard stability requirement as it appears, for example, in Shapley and Shubik (1972). The second blocking is by a principal who keeps his assigned agent but becomes strictly better off by changing the terms of the contract. The principal blocking is not standard, but it is needed in our context for two reasons. First, the transaction goes through a contract, contrary to the standard matching (firm-worker) model where the transaction goes through a price. A contract is not simply a transfer, but rather it determines the size of the surplus, and therefore a requirement must be put in place to ensure that its terms (i.e., the share α) are chosen optimally. Second, the principal side of the market has all the bargaining power. As a consequence, no equilibrium should entail a principal offering his assigned agent a payment greater than the "minimum necessary" (more on this in Sect. 4.3). Note that the "minimum necessary" may be strictly greater than the agent's reservation utility. If 1 or 2 occurs, then we say that a principal-agent pair or an individual principal block the initial outcome $(\mathcal{C}, m)$.

Definition 4 (*Principal-agent market equilibrium, PAME*) A market outcome $(\mathcal{C}, m)$ constitutes an equilibrium if: (i) it provides to all parties at least their respective reservation utilities (individually rational), (ii) it is not blocked by a pair of players or any individual principal and (iii) given a menu of contracts, effort is chosen by each agent to maximize his expected utility.

A PAME $(\mathcal{C}, m)$ achieves two goals: first, it generates the highest possible surplus in a given principal-agent partnership and guarantees that no agent is "overpaid" and second it ensures that the aggregate (market) efficiency loss due to the moral hazard impediment is minimized.[10]

3 Analysis based on an isolated principal-agent pair

In this section, we solve the problem of agent a employed by principal p . We use the results obtained here in the next section when we embed the fixed pair into the principal-agent market. When an isolated principal-agent pair is studied, the principal will never pay the agent more than the agent's reservation utility and hence there is no point in including the bonus $b_{p,a}$ in this section. The bonus will be introduced in the next section and will be the price which clears the principal-agent market. The agent chooses a level of effort (given his wage) to maximize his expected utility. That is,

$$\max_{e_a} E\left[V\right] = E[1 - \exp[-r_a(\beta_{p,a} + \alpha_{p,a}(f(e_a) + \varepsilon_p) - C(e_a))]].$$

It can be verified, using the fact that the moment generating function of a normal random variable $Y \sim N(\mu, \sigma^2)$ is $E\left[e^{tY}\right] = e^{t\mu + \frac{t^2\sigma^2}{2}}$, that the first order condition (FOC) of the above problem is,

$$\alpha_{p,a} f^{'}(e_a) - C^{'}(e_a) = 0. \tag{1}$$

[10] Our model is similar to the one in Crawford and Knoer (1981), who consider a competitive market where heterogeneous workers are matched with heterogeneous firms. In that model, a_{ij}, b_{ij} and s_{ij} is the ith worker's job satisfaction, productivity and salary at the jth firm. Thus worker i's total satisfaction at the jth firm is $a_{ij} + s_{ij}$ and the firm's profit is $b_{ij} - s_{ij}$. The two main differences are: (i) our model is more structural and (ii) it is designed to fit the specific principal-agent market, rather than an abstract firm-worker economy. As a consequence, by sacrificing on the generality, we are able to offer a sharper characterization of the problem.

Let $e_a(\alpha_{p,a})$ be agent a's optimal effort when he is employed by principal p who offers him a share of output equal to $\alpha_{p,a}$. It can be checked that,

$$\frac{de_a(\alpha_{p,a})}{d\alpha_{p,a}} = -\frac{f^{'}(e_a)}{\alpha_{p,a} f^{''}(e_a) - C^{''}(e_a)}, \tag{2}$$

which, given our assumptions, is strictly positive. Thus, as expected, the agent's effort is positively related to the share of output he gets to keep. Obviously, a fixed-rent contract (where $\alpha = 1$) provides the highest incentives for work. A fixed-rent contract however does not share the risk efficiently between the principal and the agent, as the risk averse agent is forced to bear all the risk. An optimal contract would balance incentives and risk sharing appropriately.

Now let's turn to principal p's problem. By assumption principals have all the bargaining power. Therefore the principal will choose the agent's fixed salary, $\beta_{p,a}$, such that the agent gets exactly his zero reservation utility. By plugging $e_a(\alpha_{p,a})$ into the agent's expected utility, setting it equal to his reservation payoff, and solving for $\beta_{p,a}$ we obtain,

$$\beta_{p,a} = -\alpha_{p,a} f(e_a(\alpha_{p,a})) - \alpha_{p,a}\mu_p + C(e_a(\alpha_{p,a})) + \frac{\alpha_{p,a}^2 \sigma_p^2 r_a}{2}. \tag{3}$$

Using Eq. (3), principal p's expected profit function can be written as,

$$\begin{aligned}\Pi_{p,a} &= (1-\alpha_{p,a})\left[f(e(\alpha_{p,a})) + \mu_p\right] - \beta_{p,a} \\ &= f(e(\alpha_{p,a})) + \mu_p - C(e(\alpha_{p,a})) - \frac{\alpha_{p,a}^2 \sigma_p^2 r_a}{2}.\end{aligned} \tag{4}$$

The only difference between the above profit function and $\bar{\Pi}_{p,a}$ of Sect. 2, is that the former does not include the bonus $b_{p,a}$. The principal chooses $\alpha_{p,a}$ to maximize $\Pi_{p,a}$. The FOC is,

$$\frac{d\Pi_{p,a}}{d\alpha_{p,a}} = \left[f^{'}(e(\alpha_{p,a})) - C^{'}(e(\alpha_{p,a}))\right]\frac{de_a}{d\alpha_{p,a}} - \alpha_{p,a}\sigma_p^2 r_a = 0,$$

which, by using Eq. (1) (Envelope Theorem), becomes,

$$\frac{d\Pi_{p,a}}{d\alpha_{p,a}} = (1-\alpha_{p,a}) f^{'}(e(\alpha_{p,a}))\frac{de_a}{d\alpha_{p,a}} - \alpha_{p,a}\sigma_p^2 r_a = 0. \tag{5}$$

The principal's profit function is strictly concave in $\alpha_{p,a}$. This can be seen by computing the second derivative,

$$\frac{d^2\Pi_{p,a}}{d\alpha_{p,a}^2} = -f^{'}(e(\alpha_{p,a}))\frac{de_a}{d\alpha_{p,a}} + (1-\alpha_{p,a})f^{''}(e(\alpha_{p,a}))\left(\frac{de_a}{d\alpha_{p,a}}\right)^2 + (1-\alpha_{p,a})f^{'}(e(\alpha_{p,a}))\frac{d^2e_a}{d\alpha_{p,a}^2} - \sigma_p^2 r_a, \quad (6)$$

and checking that, under our assumptions, it is strictly negative. The first, second and fourth terms on the right hand side of Eq. (6) are clearly negative. The third term can be derived by differentiating Eq. (2) with respect to $\alpha_{p,a}$,

$$\frac{d^2e_a}{d\alpha_{p,a}^2} = -\left[\frac{f^{''}\frac{de}{d\alpha}\left(\alpha_{p,a}f^{''}(e_a) - C^{''}(e_a)\right) - f^{'}\left(f^{''} + \alpha f^{'''}\frac{de}{d\alpha} - C^{'''}\frac{de}{d\alpha}\right)}{\left(\alpha_{p,a}f^{''}(e_a) - C^{''}(e_a)\right)^2}\right].$$

We assume that the above second derivative is negative and consequently Eq. (6) is strictly negative.[11] Hence, there exists a unique $\alpha_{p,a}(r_a, \sigma_p^2) \in (0, 1)$ that solves Eq. (5). The vector of the comparative statics, by invoking the Implicit Function Theorem, is,

$$\left(\frac{\partial\alpha_{p,a}(r_a,\sigma_p^2)}{\partial r_a}, \frac{\partial\alpha_{p,a}(r_a,\sigma_p^2)}{\partial\sigma_p^2}\right) = \frac{1}{\left(d^2\Pi_{p,a}/d\alpha_{p,a}^2\right)}\left(\alpha_{p,a}\sigma_p^2,\ \alpha_{p,a}r_a\right) = \left(<0,\ <0\right). \quad (7)$$

Conditional on a matching m the outcome described so far in this section is efficient (in the constrained second best sense, due to the presence of moral hazard). The expected profit of principal p who has hired agent a is obtained by substituting $\alpha_{p,a}(r_a, \sigma_p^2)$ into Eq. (4) and is given by,

$$\Pi_{p,a}(r_a, \sigma_p^2) = f(e(\alpha_{p,a}(r_a, \sigma_p^2))) + \mu_p - C(e(\alpha_{p,a}(r_a, \sigma_p^2))) - \frac{\left[\alpha_{p,a}(r_a, \sigma_p^2)\right]^2 \sigma_p^2 r_a}{2}. \quad (8)$$

We assume that $\Pi_{p,a}(r_a, \sigma_p^2) > 0$, ruling out uninteresting cases.

[11] This is true for many (widely used) production and cost functions. For example: 1) $f = e^b$ and $C = e^c/c$, (with $b \le 1$ and $c > 1$) provided that $c \ge 1 + b$ and (2) $f = \ln(1 + e)$ and $C = e^2/2$. In case 1 the effort function is, $e^* = (\alpha b)^{1/(c-b)}$ and in case 2 $e^* = -0.5 + 0.5\sqrt{1 + 4\alpha}$. In both cases, $d^2e/d\alpha^2 \le 0$.

Next, we differentiate the expected profit function with respect to the degree of risk aversion, which yields,

$$\frac{\partial \Pi_{p,a}(r_a,\sigma_p^2)}{\partial r_a} = \left(f^{'} - C^{'}\right)\frac{de_a}{d\alpha_{p,a}}\frac{\partial \alpha_{p,a}}{\partial r_a} - \alpha_{p,a}\frac{\partial \alpha_{p,a}}{\partial r_a}\sigma_p^2 r_a - \frac{\alpha_{p,a}^2\sigma_p^2}{2}$$
$$= -\frac{\left[\alpha_{p,a}(r_a,\sigma_p^2)\right]^2\sigma_p^2}{2} < 0, \tag{9}$$

where the second equality follows from the Envelope Theorem. The expected profit function is decreasing in the agent's degree of risk aversion. A relatively low risk averse agent can tolerate more risk and therefore the principal can increase the incentives for effort (α) which increases the expected output and the principal's profit. We can also show that $\partial \Pi_{p,a}(r_a,\sigma_p^2)/\partial\sigma_p^2 < 0$. Hence, the productivity of a principal (agent) is inversely related to the variance of the asset that he owns (degree of risk aversion). Therefore, we should expect principals (who make the offers) to compete with each other for low risk averse agents.

The question which naturally arises is who ends up hiring low risk averse agents? Before we attempt to answer this question, it is apparent that only the P agents with the lowest degrees of risk aversion will be hired by the P principals. The remaining $A - P$ agents with relatively higher degrees of risk aversion do not get hired by any principal and they receive their reservation payoffs. Therefore, in what follows we assume that the number of principals is equal to the number of agents and equal to n, i.e., $P = A = n$.[12]

For example, if $f(e) = e$ and $C(e) = e^2/2$, it can be easily computed that the equilibrium level of incentives that principal p offers to agent a is given by,

$$\alpha_{p,a} = \frac{1}{1 + r_a\sigma_p^2}. \tag{10}$$

The principal's expected profit function is given by,

$$\Pi_{p,a} = \frac{1 + 2\mu_p + 2\mu_p r_a\sigma_p^2}{2(1 + r_a\sigma_p^2)}. \tag{11}$$

Next, we look for a PAME in an $n \times n$ principal-agent market.

[12] This is not to say that the presence of unmatched agents has no effect on the equilibrium outcome and in particular on the distribution of the surplus among the matched players. Nevertheless, under the minimum price competitive equilibrium concept that we will introduce in the next section, the agent with the highest degree of risk aversion among the matched agents receives a zero bonus. Moreover, each principal, in equilibrium, (weakly) prefers his assigned agent to the one with the highest degree of risk aversion among all the matched agents. Therefore, the unmatched agents (who have even higher degrees of risk aversion) become inessential as, clearly, no principal would prefer any of them over the one that he is optimally assigned to. This assertion, however, is not necessarily true in a competitive equilibrium other than the minimum price one.

4 Analysis of a principal-agent market

In this section we proceed as follows. First, using Shapley and Shubik's (1972) assignment game, we characterize the set of optimal matchings. We show that any PAME outcome must involve an optimal assignment between the principals and the agents. In general, such an assignment involves (in our setting) both a negatively and a positively assortative matching (see Sect.4.24.2 for a formal definition). Then, and given an optimal assignment, we find a price which supports that assignment as a PAME. Since the principals have all the bargaining, we argue that the focal point among all PAME outcomes should be the minimum price competitive equilibrium (e.g. Demange et al. 1986; Leonard 1983), where no agent is overpaid. This is the most favorable outcome for the principal side of the market. Finally, we characterize the main properties of an equilibrium. For the sake of completeness, in the next subsection, we review some well-established results.

4.1 Preliminaries (The assignment game)

The PAME concept that we imposed is closely related to the set of optimal assignments in the $n \times n$ game. Thus, we begin by studying the assignment game.[13] There are two finite disjoint sets of players P and A where each set contains n players. Principals are in the P set and agents in the A set. Each principal is matched with exactly one agent.

The Pareto frontiers associated with each principal-agent pair: (i) are continuous, (ii) are strictly decreasing, (iii) have the point $(0, 0)$ as their origin (principals' and agents' reservation payoffs) and (iv) are non-linear (since the agents' utility functions are non-linear). Clearly, a principal's offer must always lie on the Pareto frontier. In other words, α (share) must be chosen optimally and β (fixed salary) must be set at the level which yields zero expected utility to the agents [see Sect. 3 where (α, β) are chosen]. A re-parameterization of the utility possibility frontiers to linear ones of the form, $u_p + b_a = \Pi_{p,a}$, for all (p, a) in $P \times A$, is *inessential* [see Crawford and Knoer (1981, p.448) for a discussion on this point]. Therefore, associated with each possible matching (p, a) in $P \times A$ is a nonnegative real number $\Pi_{p,a}$ which denotes principal p's maximized profits if he employs agent a (see Eq. (8)). A game in coalitional form with side payments is determined by (P, A, Π), with the numbers $\Pi_{p,a}$ being the worth of the coalitions (p, a) consisting of one principal and one agent. Any pair of players (p, a) in $P \times A$ can together obtain $\Pi_{p,a}$ and any larger coalition is valuable as long as it can break into pairs. The members of any coalition can divide among themselves the collective worth of the coalition in any way they wish. The coalitional function v is given by,

$$v(S) = \Pi_{p,a} \text{ if } S = \{p, a\} \text{ for } p \in P \text{ and } a \in A;$$
$$v(S) = 0 \text{ if } S \text{ contains only } P - \text{principals or only } A - \text{agents};$$

[13] This subsection borrows notations and definitions from Roth and Sotomayor (1990, Chap. 8).

$v(S) = \max\left(v(p_1, a_1) + v(p_2, a_2) + \cdots + v(p_k, a_k)\right)$ for arbitrary coalitions S, with the maximum to be taken over all sets $\{(p_1, a_1), \ldots, (p_k, a_k)\}$ of k distinct pairs in $S_P \times S_A$, where S_P and S_A denote the sets of P – principals and A – agents in S.

The evaluation of the maximization problem to determine $v(S)$ for a given matrix Π is called an *assignment problem*. In particular, we are interested in the value of the coalition $P \cup A$ since $v(P \cup A)$ equals the maximum total payoff available to the players in this game. Consider the following linear programming (LP) problem:

$$\begin{array}{l}\underset{x}{\text{Maximize}} \sum_{p,a} \Pi_{p,a} \cdot x_{p,a} \\ \text{subject to :} \\ (a)\ \sum_p x_{p,a} \leq 1 \\ (b)\ \sum_a x_{p,a} \leq 1 \\ (c)\ x_{p,a} \geq 0. \end{array} \tag{LP1}$$

It is a well-known result (e.g. Roth and Sotomayor 1990, Chap. 8) that there exists a solution to this LP problem which involves only values of zero and one. Then, $v(P \cup A) = \sum_{p,a} \Pi_{p,a} \cdot x_{p,a}$, where x (an $n \times n$ matrix with zeroes and ones) is an optimal solution (assignment) to the LP problem.

Lemma 1 *If $(\mathcal{C}, m)$ is a PAME, then the assignment must be optimal.*

Proof See appendix □

Hence, in search of a PAME, we can restrict ourselves to the set of optimal assignments. Then, we need to find a set of prices (bonuses) which will support an optimal assignment as a PAME outcome. This will be the minimum price competitive equilibrium which is defined below.

A multi-item auction mechanism Demange et al. (1986) The set of principals P is the set of bidders and the set of agents A is the set of objects (each set contains n elements). Each object has a reservation price of zero. The value of object a to principal p is $\Pi_{p,a}$. A feasible bonus (price) vector $\mathbf{b}$ is a function from A to R_+ such $b_a \geq 0$ for all $a \in A$. The demand set of a principal p at bonus $\mathbf{b} = (b_a)_{a \in A}$ is defined by,

$$D_p(\mathbf{b}) = \left\{ a \in A : \Pi_{p,a} - b_a = \max_{a' \in A} \left\{ \Pi_{p,a'} - b_{a'} \right\} \right\}.$$

The contract vector $\mathcal{C}$ (see Sect. 2 where $\mathcal{C}$ is defined) is called *quasi-competitive* if there is a matching m from P to A such that if $m(p) = a$ then $a \in D_p(\mathbf{b})$. The matching m is said to be compatible with the vector $\mathcal{C}$.

Definition 5 (*Competitive equilibrium, CE*) The pair $(\mathcal{C}, m)$ is a competitive equilibrium if $\mathcal{C}$ is quasi-competitive and m is compatible with $\mathcal{C}$.

Definition 6 (*Minimum price competitive equilibrium, MPCE*) The pair $(\mathcal{C}, m)$ is a minimum price competitive equilibrium if in addition to being a CE, the bonus vector, $\mathbf{b}_{\min}$, is at least as small in every component as any other CE bonus vector.

Now observe that if $(\mathcal{C}, m)$ is a MPCE, then it is also a PAME.[14] The reverse, however, is not necessarily true (more on this in Sect. 4.3). Moreover, a CE may not be a PAME, as a CE is not necessarily immune from unilateral principal deviations (Definition 3). Hence, in general, we have the following relationship among the three equilibrium concepts that we have introduced so far,

$$MPCE \subseteq PAME \subseteq CE.$$

As we show in Sect. 4.3, under some conditions, the MPCE coincides with the PAME. Finally, even when the MPCE does not coincide with the PAME, one can specify a mechanism through which contracts are offered and adjusted. This *contract adjustment* process will converge to a MPCE. For these reasons, we focus on the MPCE which we also know (e.g. Roth and Sotomayor 1990, Chap. 8) it exists (see Sect. 4.3 for more details).

Therefore, in the next two sections, we set out to describe how one can compute a MPCE. We begin, in the next section, by probing deeper into the properties of an optimal assignment, without any reference to the issue of surplus distribution among players. The latter is being addressed in Sect. 4.3 where we show how a MPCE can be derived.

4.2 Characterization of an optimal matching

Let m^* be an optimal assignment (matching). The matching function $m^* : P \rightarrow A$ defines the optimally assigned principal-agent pairs. Each principal $p \in P$ is identified by the variance of his asset σ_p^2 and each agent $a \in A$ by his degree of risk aversion r_a.

Definition 7 (*Matching curve*) A matching curve is the locus of (variance-risk aversion) pairs induced by the matching function m^* (σ_p^2 are on the horizontal axis and r_a on the vertical).

A principal's (agent's) productivity is in inverse relationship with the variance of his asset (his degree of risk aversion).

Definition 8 (*Assortative matching*) A matching is: (1) positively assortative (PAM) if high productivity principals are matched with high productivity agents (the matching curve has a positive slope) and (2) negatively assortative (NAM) if high productivity principals are matched with low productivity agents (the matching curve has a negative slope).

[14] First, by construction of $\mathcal{C}$ all players receive at least their reservation utilities. Second, no principal p would wish to contract with an agent other than the one that is currently assigned to, and is in his demand set $D_p(\mathbf{b}_{\min})$, ruling out the possibility of a pair blocking. Finally, no principal p would offer a new contract to his assigned agent a. The level of incentives $\alpha_{p,a}$ is chosen optimally (see Sect. 3) and therefore no change is warranted. The fixed salary $\beta_{p,a}$ is set to ensure that the participation constraint is met and hence it cannot be changed either. Lastly, since $\mathbf{b}_{\min}$ is the minimum equilibrium price vector, it cannot be lowered unilaterally by any principal without him loosing his assigned agent, ruling out the possibility of an individual (principal) blocking.

Definition 9 (*Submodularity and supermodularity*) The profit function $\Pi(r, \sigma^2)$ is submodular if: $r > r^{'}$ and $\sigma^2 > \sigma^{2^{'}}$, then $\Pi(r, \sigma^{2^{'}}) + \Pi(r^{'}, \sigma^2) > \Pi(r^{'}, \sigma^{2^{'}}) + \Pi(r, \sigma^2)$ and supermodular if: $r > r^{'}$ and $\sigma^2 > \sigma^{2^{'}}$, then $\Pi(r^{'}, \sigma^{2^{'}}) + \Pi(r, \sigma^2) > \Pi(r^{'}, \sigma^2) + \Pi(r, \sigma^{2^{'}})$.

If the profit function is supermodular, then all principals have higher marginal products when they match with high productivity agents, i.e., the traits of the principals and the agents (variance and degree of risk aversion) are complements. If the profit function is submodular, then all principals have higher marginal products when they match with low productivity agents, i.e., the traits are substitutes. The literature on assortative matching assumes that the production function is either supermodular or submodular (e.g. Shimer and Smith 2000).[15] In our paper we cannot make such an assumption since our production function (profit function) is not part of the data of our problem, but rather the outcome of principal-agent interaction. Therefore, we have to find conditions under which $\Pi(r, \sigma^2)$ is supermodular, submodular, or both. This is what we do next.

Consider any two agents j and k with $r_j < r_k$. Let $M^{(j-k)}(\sigma_p^2) = \Pi_{p,j}(\sigma_p^2) - \Pi_{p,k}(\sigma_p^2)$ denote principal p's incremental profits when he hires agent j instead of agent k. The next Lemma characterizes the $M^{(j-k)}(\sigma_p^2)$ function. The properties of this function will then be used to characterize $\Pi(r, \sigma^2)$.

Lemma 2 $M^{(\cdot,\cdot)}(\cdot)$ *has the following properties:*

(1) $M^{(j-k)}(\sigma_p^2)$ *is positive for any* $\sigma_p^2 \geq 0$,
(2) $M^{(j-k)}(\sigma_p^2) \geq M^{(j^{'}-k)}(\sigma_p^2)$ *for all* σ_p^2, *if* $r_{j^{'}} > r_j$ *and* $M^{(j-k^{'})}(\sigma_p^2) \geq M^{(j-k)}(\sigma_p^2)$ *for all* σ_p^2, *if* $r_{k^{'}} < r_k$,
(3) $M^{(j-k)}(0) = 0$,
(4) *There exists a threshold* $\hat{\sigma}^2(r_j, r_k)$, *such that: a) for any* $\sigma_p^2 < \hat{\sigma}^2(r_j, r_k)$, $\partial M^{(j-k)}(\sigma_p^2)/\partial\sigma_p^2 > 0$, *b) for any* $\sigma_p^2 > \hat{\sigma}^2(r_j, r_k)$, $\partial M^{(j-k)}(\sigma_p^2)/\partial\sigma_p^2 < 0$ *and c)* $\partial M^{(j-k)}(\hat{\sigma}^2)/\partial\sigma_p^2 = 0$,
(5) *If we fix* r_j *and decrease* r_k, *or fix* r_k *and decrease* r_j, $\hat{\sigma}^2(r_j, r_k)$ *will increase, i.e.,*

$$\hat{\sigma}^2(r_j, r_{k^{'}}) > \hat{\sigma}^2(r_j, r_k), \quad \textit{if}\; r_{k^{'}} < r_k,$$
$$\hat{\sigma}^2(r_{j^{'}}, r_k) > \hat{\sigma}^2(r_j, r_k), \quad \textit{if}\; r_{j^{'}} < r_j.$$

Proof See Appendix. □

It can be seen (from the proof of Lemma 2) that $M^{(j-k)}(\sigma_p^2)$ does not depend on the mean μ_p of the stochastic component ε_p. The incremental profit function $M^{(j-k)}(\cdot)$ exhibits an inverse U-shape. This implies that each principal's willingness to pay for

[15] One exception is Anderson and Smith (2003) who also (as we do) derive the production function, instead of assuming its properties.

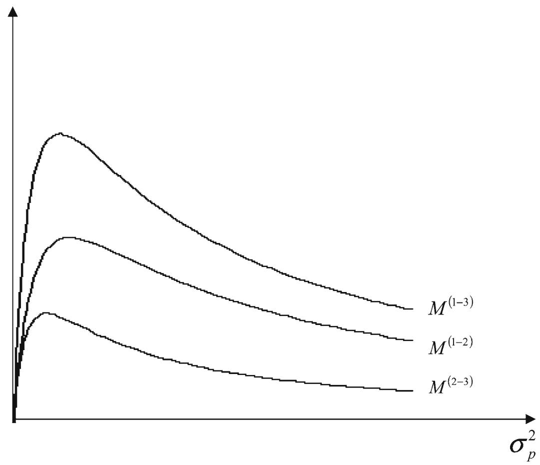

Fig. 1 $M^{(j-k)}(\sigma_p^2)$functions when $f(e) = e, C(e) = e^2/2$ and $\mathbf{r} = (1, 2, 3)$

a lower risk averse agent is very low when the principal's technology is subject to a very low or very high risk. For moderate values of risk the willingness to pay is high. *Example*. When $f(e) = e$ and $C(e) = e^2/2$, then,

$$M^{(j-k)}(\sigma_p^2) = \frac{\sigma_p^2(r_k - r_j)}{2(1 + r_j\sigma_p^2)(1 + r_k\sigma_p^2)} \geq 0,$$

and $\hat{\sigma}^2(r_j, r_k) = 1/\sqrt{r_j r_k}$. Notice that $M^{(j-k)}(0) = 0$, $\lim_{\sigma^2\to\infty} M^{(j-k)}(\sigma_p^2) = 0$ and $M^{(j-k)}(\sigma_p^2)$ attains its maximum at $\hat{\sigma}^2(r_j, r_k) = 1/\sqrt{r_j r_k}$.

Figure 1 depicts the incremental profit functions when there are three agents with degrees of risk aversion 1, 2 and 3 respectively. From Lemma 2 part 2) we know that $M^{(1-3)}(\sigma_p^2) \geq M^{(1-2)}(\sigma_p^2)$ and $M^{(1-3)}(\sigma_p^2) \geq M^{(2-3)}(\sigma_p^2)$, which is verified by the numerical example. Also in this numerical example, $M^{(1-2)}(\sigma_p^2) \geq M^{(2-3)}(\sigma_p^2)$. Overall, we have,

$$M^{(1-3)}(\sigma_p^2) \geq M^{(1-2)}(\sigma_p^2) \geq M^{(2-3)}(\sigma_p^2).$$

Moreover,

$$\hat{\sigma}^2(1, 2) = \frac{1}{\sqrt{2}} \approx 0.71 > \hat{\sigma}^2(1, 3) = \frac{1}{\sqrt{3}} \approx 0.58 > \hat{\sigma}^2(2, 3) = \frac{1}{\sqrt{6}} \approx 0.4.$$

Corollary 1 *The profit function $\Pi(r, \sigma^2)$ is supermodular for all the σ^2's that are greater than $\hat{\sigma}^2(r_1, r_2)$ and submodular for all the σ^2's that are less than $\hat{\sigma}^2(r_{n-1}, r_n)$.*

Proof First we show that,

$$\hat{\sigma}^2(r_{n-1}, r_n) < \hat{\sigma}^2(r_1, r_2).$$

This is true since, from part 5) of Lemma 2, $\hat{\sigma}^2(r_{n-1}, r_n) < \hat{\sigma}^2(r_1, r_n)$ and $\hat{\sigma}^2(r_1, r_n) < \hat{\sigma}^2(r_1, r_2)$. Thus, for all $\sigma^2 < \hat{\sigma}^2(r_{n-1}, r_n)$, all the $M^{(\cdot,\cdot)}(\cdot)$ functions are increasing. Let $\sigma^2 > \sigma^{2'}$ and both lower than $\hat{\sigma}^2(r_{n-1}, r_n)$ and $r > r'$. Then, $M^{(r'-r)}(\sigma^2) > M^{(r'-r)}(\sigma^{2'})$ for any $r, r' \in \{r_1, \ldots, r_n\}$. This implies,

$$\Pi(r', \sigma^2) - \Pi(r, \sigma^2) > \Pi(r', \sigma^{2'}) - \Pi(r, \sigma^{2'})$$
$$\Rightarrow \Pi(r', \sigma^2) + \Pi(r, \sigma^{2'}) > \Pi(r', \sigma^{2'}) + \Pi(r, \sigma^2),$$

and the profit function is submodular. We can similarly prove that for all $\sigma^2 > \hat{\sigma}^2(r_1, r_2)$, the profit function is supermodular. □

The two thresholds $\hat{\sigma}^2(r_{n-1}, r_n)$ and $\hat{\sigma}^2(r_1, r_2)$ divide the $[0, \infty)$ interval, where σ^2's lie, into three regions. The size of each region depends on the vector $\mathbf{r}$ and in particular on r_1, r_2, r_{n-1} and r_n. We list the following three additional assumptions (which will not hold simultaneously) regarding the vectors $\mathbf{r}$ and $\boldsymbol{\sigma}^2$.

Assumption A The lowest variance, σ_1^2, is greater than $\hat{\sigma}^2\,(r_1, r_2)$. This can be true in markets where either the agents have high degrees of risk aversion (and hence $\hat{\sigma}^2\,(r_1, r_2)$ is relatively low), or the level of variances is high (and hence σ_1^2 is relatively high).

Assumption B The highest variance, σ_n^2, is less than $\hat{\sigma}^2\,(r_{n-1}, r_n)$. This can be true in markets where either the agents have low degrees of risk aversion (and hence $\hat{\sigma}^2\,(r_{n-1}, r_n)$ is relatively high), or the level of variances is low (and hence σ_n^2 is relatively low).

Assumption C The lowest variance, σ_1^2, is less than $\hat{\sigma}^2\,(r_1, r_2)$ and the highest variance, σ_n^2, is greater than $\hat{\sigma}^2\,(r_{n-1}, r_n)$. This can be true in markets where the distribution of the variances is relatively very dispersed.

The next Theorem characterizes the properties of an optimal assignment between principals and agents.

Theorem 1 *Consider a market with n principals and n agents which satisfies the assumptions of our model. The optimal assignment is,*

1. *PAM, if assumption A is satisfied,*
2. *NAM, if assumption B is satisfied*
3. *simultaneously negatively and positively assortative (not globally assortative, NGAM), if assumption C is satisfied.*

Proof Under assumption A the profit function is supermodular, while under assumption B it is submodular (Corollary 1). Given these results, the rest of the proof can be based on a well-known result by Becker (1973) [see also Shimer and Smith 2000] and therefore is omitted. □

The nature of the optimal assignment critically depends on the properties of the two vectors $\mathbf{r}$ and $\boldsymbol{\sigma}^2$. An immediate Corollary to Theorem 1 is that for all σ^2's less than $\hat{\sigma}^2(r_{n-1}, r_n)$ the matching is negatively assortative, while for all σ^2's greater than $\hat{\sigma}^2(r_1, r_2)$ it is positively assortative. Note that the structure of our problem does not allow us to characterize the matching properties for the σ^2's that lie in the interval $\left[\hat{\sigma}^2(r_{n-1}, r_n), \hat{\sigma}^2(r_1, r_2)\right]$.[16]

If all agents had a zero degree of risk aversion then the matching problem would be trivial. In this case, any matching is optimal (first-best) since the outcome of any principal-agent relationship is efficient, due to the absence of the trade-off between incentives and insurance. When such a trade-off is present, however, (i.e., when $r_a > 0$ for all $a \in A$), then an optimal assignment maximizes the sum of all principals' profits, or equivalently minimizes the distance from the first best profits. For relatively low variances the traits of the principals (variance) and the agents (degree of risk aversion) are *substitutes*. In this case the loss of efficiency is minimized when a low variance principal is matched with a high risk averse agent. The intuition is best illustrated if we consider an example with two principals and two agents. Since variances are low, the lower variance principal is very close to the efficient frontier and therefore does not gain as much as the higher variance principal would by employing the lower risk aversion agent. When the variances are relatively high, then the traits are *complements*. In this case the intuition works in the opposite direction. The higher variance principal is very far from the efficient frontier and hiring the lower risk aversion agent is not going to increase his profits as much as it would if this agent was hired by the lower variance principal. The reason for the possible co-existence of substitutable and complementary traits is that the profit function $\Pi(r, \sigma^2)$, in general, exhibits *non-monotonic marginal products*.

In the next section we describe how, based on an optimal assignment, we can compute a MPCE.

4.3 Computing the minimum price competitive equilibrium (MPCE)

Consider an optimal assignment m^*. Given this assignment we can now compute the minimum competitive bonus vector $\mathbf{b}^*_{\min}$ as follows. There are two distinct cases that we will consider.

Case 1 (*Assumption A or B is satisfied*). Let's begin by assuming that assumption A holds, i.e., the matching is positively assortative. The argument when assumption B (i.e., negatively assortative matching) holds will be similar and it is omitted. Positively assortative matching means that low risk principals are matched with low degree of risk aversion agents. So principal 1 is matched with agent 1, principal 2 with agent 2

[16] When the production function (profit function in our case) exhibits non-monotonic marginal products (like in our model), general matching patterns are not easily characterized, [see Shimer and Smith 2000, p.346)]. Also, the size of the interval $\left[\hat{\sigma}^2(r_{n-1}, r_n), \hat{\sigma}^2(r_1, r_2)\right]$ depends on how wide the support of the distribution of the degrees or risk aversion in a given market is. In many markets (e.g. market for CEOs), as our theory predicts, the upper tail of the distribution (where the very risk averse agents are) gets truncated since individuals with high degrees of risk aversion will not be hired to run risky projects and consequently r_n is relatively close to r_1 and the size of $\left[\hat{\sigma}^2(r_{n-1}, r_n), \hat{\sigma}^2(r_1, r_2)\right]$ becomes small. Nevertheless, given a specific numerical problem, the optimal matching can be completely characterized by solving the LP1 problem (see Example 3 in Sect. 6).

and so on (recall that principal and agent 1 have the lowest variance and degree of risk aversion, respectively, while principal n and agent n the highest). First it is clear that the agent with the highest degree of risk aversion will receive a zero bonus, i.e., $b^*_{n,\min} = 0$. Next, principal $n-1$ who is matched with agent $n-1$ (i.e., $m^*(n-1) = n-1$) offers his agent a bonus equal to $b^*_{n-1,\min} = \Pi_{n,n-1} - \Pi_{n,n} + b^*_{n,\min}$. That is, a bonus equal to principal n's incremental profit if he employed agent $n-1$ instead of agent n. We can proceed this way until $b^*_{1,\min}$. Along this way, each principal (say principal $l-1$) pays his assigned agent ($m^*(l-1) = l-1$) a bonus $b^*_{l-1,\min}$ equal to the incremental profit $\Pi_{l,l-1} - \Pi_{l,l} + b^*_{l,\min}$ of principal l who instead of his assigned agent l hires the agent with the next lower degree of risk aversion, i.e., agent $l-1$.

Under assumption A or B the competition among principals for agents with low degrees of risk aversion has a great deal of structure (because all the variance-risk aversion pairs are positioned either on the increasing or on the decreasing portions of the $M^{(\cdot,\cdot)}(\cdot,\cdot)$ functions; see proof of the Proposition below). In particular, each principal competes directly only with the principal that has the next higher variance asset. This allows us to offer the above simple algorithm that finds a MPCE (see Examples 1 and 2 in Sect. 6).

Proposition 1 *Suppose that either assumption A or B is satisfied and let m^* be an optimal assignment. Moreover $\left(\alpha_{p,a}, \beta_{p,a}\right)_{p\in P, a\in A}$ are computed optimally for each (p, a) as shown in Sect. 3 and $\left(b_{p,a}\right)_{p\in P, m^*(p)\in A}$ is computed using the above simple algorithm. Then, the resulting $(\mathcal{C}^*, m^*)$ is a MPCE.*

Proof See appendix □

Case 2 (*Assumption C is satisfied*). In this case the simple algorithm described above does not necessarily lead to a MPCE. To find the MPCE, we will resort to the following general linear programming problem (e.g. Leonard 1983, p. 473).

$$
\begin{aligned}
&\text{Minimize } \textstyle\sum_a b_a \\
&\text{subject to:} \\
&(a)\ u_p + b_a \geq \Pi_{p,a} \\
&(b)\ u_p, b_a \geq 0 \\
&(c)\ \textstyle\sum_p u_p + \sum_a b_a = V
\end{aligned}
\tag{LP2}
$$

where V is the value of LP1. Note that LP2 finds the lowest bonuses (prices) in the class of bonuses which meet the other conditions of the dual to LP1 problem and which attain LP1's optimum value (see Example 3 in Sect. 6). The dual to LP1 problem is, min $\sum_p u_p + \sum_a b_a$ subject to (a) and (b) of LP2. The solutions to the dual to LP1 problem is the set of CE points of the game which coincide with the core, in absence of the no-principal blocking requirement of Definition 3 (e.g. Shapley and Shubik 1972). LP2 finds the CE at which every principal gets the maximum payoff, while every agent gets the minimum payoff (i.e., the MPCE). Clearly, the core of our game which coincides with the set of PAME points is a subset of the set of solutions to the dual to LP1 problem.

The next Proposition sheds more light on the relationship between the set of PAME and the MPCE and CE outcomes.

Proposition 2 *Suppose that either assumption A or B hold. If* $(\mathcal{C}, m)$ *is a PAME, then it must also be a MPCE.*

Proof See appendix □

In addition, a CE may not be a PAME. This can be easily seen from Example 1 of Sect. 6, where assumption A holds. The minimum bonus vector $\mathbf{b}_{\min}$ along with $\left(\alpha_{p,a}, \beta_{p,a}\right)$ and m constitute a MPCE and hence a PAME. Now construct a new bonus vector as follows, $\mathbf{b} = \mathbf{b}_{\min} + \mathbf{c}$, with $\mathbf{c} = (c, \ldots, c) \gg 0$. The new outcome is clearly a CE, but not a PAME since principal 4, who has employed the agent with the highest degree of risk aversion, can unilaterally deviate by slightly lowering b_4.

We have already shown in Sect. 4.1 that $MPCE \subseteq PAME$. Hence, Proposition 2 and the discussion that proceeds it imply that under assumption A or B,

$$MPCE = PAME \subset CE.$$

This result is interesting because it ensures that when one side of the market has all the bargaining power, then the MPCE is the *unique* outcome. The above assertion is not necessarily true when neither one of these two assumptions (A or B) is satisfied.[17] Interestingly, this suggests that the MPCE is not always guaranteed to the side of the market who has all the bargaining power. To arrive at the MPCE, in general, one has to further specify a dynamic process by which contracts are adjusted to clear the market. This process will mimic the *salary adjustment* algorithm (e.g. Crawford and Knoer 1981), which is similar in spirit to the *deferred acceptance* procedure introduced by Gale and Shapley (1962) in the case of non-transferable utility.[18] These algorithms can be justified when one side of the market has all the bargaining power and hence it makes the offers, as it is the case in our $n \times n$ principal-agent market.

Finally, it can be easily seen that in *any* PAME the bonus that an agent receives is in inverse relationship with his degree of risk aversion.

In the next section, we attempt to relate our theoretical findings with some empirical observations.

5 Relationship between risk $\left(\sigma^2\right)$ and: (i) incentives (α), (ii) degree of risk aversion (r) and (iii) fixed compensation (B)

The vast majority of the empirical literature has assumed that the representative entity is a fixed principal-agent pair. Under this assumption both r and σ^2 are exogenous and the equation which maps these two parameters into the endogenous variable is: $\alpha = \alpha(r, \sigma^2)$, with $\partial\alpha/\partial r < 0$ and $\partial\alpha/\partial\sigma^2 < 0$ [see Eq. (7)]. In this article, we argue that the representative entity is not a pair, but the entire $n \times n$ principal-agent

[17] We have an example (under assumption C) which demonstrates that a bonus vector could be greater than $\mathbf{b}_{\min}$ and nevertheless the resulting outcome is a PAME. More specifically, no principal has a unilateral incentive to lower the bonus he offers to his assigned agent. The example is available upon request.

[18] For these algorithms to work, the bonus (price) must not be continuous, but rather discrete. Nevertheless, by defining arbitrarily small incremental changes in the bonus, we can approximate the continuous outcome well (see Crawford and Knoer (1981, p. 448).

market. In addition, the MPCE constitutes a sensible solution concept. We derive two equations which characterize an equilibrium relationship,

$$\alpha = \alpha(r, \sigma^2) \quad \text{and} \quad r = m(\sigma^2),$$

or,

$$\alpha = \alpha(m(\sigma^2), \sigma^2) \quad \text{and} \quad r = m(\sigma^2). \tag{12}$$

Equation (12) indicates how the exogenous parameter σ^2 affects the two endogenous variables (α, r). Suppose that σ^2 changes (from $\sigma^{2'}$ to $\sigma^{2''}$) and denote this change by $\Delta\sigma^2 = \sigma^{2''} - \sigma^{2'}$. Furthermore, denote by $\Delta r = r'' - r' = m(\sigma^{2''}) - m(\sigma^{2'})$ the change in the degree of risk aversion along the matching curve and by $\Delta\alpha = \alpha(m(\sigma^{2''}), \sigma^{2''}) - \alpha(m(\sigma^{2'}), \sigma^{2'})$ the total change in the level of incentives. We will decompose $\Delta\alpha$ into two parts, the first being the change in incentives when the degree of risk aversion is kept unchanged,

$$\Delta\alpha\left(\sigma^2 | r = r'\right) = \alpha(r', \sigma^{2''}) - \alpha(r', \sigma^{2'}) = \int\limits_{\sigma^{2'}}^{\sigma^{2''}} \frac{\partial\alpha(r', \sigma^2)}{\partial\sigma^2} d\sigma^2$$

$$= \text{using Eq. (7)} = \int\limits_{\sigma^{2'}}^{\sigma^{2''}} \frac{\alpha(r', \sigma^2) r'}{d^2\Pi/d\alpha^2} d\sigma^2, \tag{13}$$

and the second being the change in incentives when the degree of risk aversion is allowed to move along the matching curve,

$$\Delta\alpha(r|\sigma^2 = \sigma^{2''}) = \alpha(\left[m(\sigma^{2'}) + \Delta r\right], \sigma^{2''}) - \alpha(m(\sigma^{2'}), \sigma^{2''}) = \int\limits_{r'}^{r''} \frac{\partial\alpha(r, \sigma^{2''})}{\partial r} dr$$

$$= \text{using Eq. (7)} = \int\limits_{r'}^{r''} \frac{\alpha(r, \sigma^{2''})\sigma^{2''}}{d^2\Pi/d\alpha^2} dr. \tag{14}$$

The total effect of risk (rates of change) on the level of incentives is given by,

$$\frac{\Delta\alpha}{\Delta\sigma^2} = \underset{(-)}{\frac{\Delta\alpha\left(\sigma^2 | r = r'\right)}{\Delta\sigma^2}} + \underset{(+,-)}{\frac{\Delta\alpha(r|\sigma^2 = \sigma^{2''})}{\Delta\sigma^2}} \lesseqqgtr 0, \tag{15}$$

which could be either positive or negative depending upon the sign and magnitude of Δr.

An increase in σ^2 has two effects on the level of incentives: a direct effect [first term of Eq. (15)] and an indirect effect [second term of Eq. (15)]. The direct effect is always negative since holding r fixed an increase in variance lowers incentives. The indirect term could be either positive or negative depending on the slope of the matching curve $\Delta r/\Delta\sigma^2$. We can characterize the incentives function $\alpha(m(\sigma^2), \sigma^2)$ by comparing these two effects, which results in the following three cases:

1. Suppose assumption B holds. The matching curve is everywhere negatively sloped (negatively assortative matching), i.e., $\Delta r/\Delta\sigma^2 < 0$. This implies that the second term in Eq. (15) is positive. Hence the overall effect of risk on incentives depends on the relative strength of the two terms in Eq. (15). If the matching curve is sufficiently steep, then $|\Delta r|$ is sufficiently greater than $\left|\Delta\sigma^2\right|$, which implies that $\left|\Delta\alpha(r|\sigma^2 = \sigma^{2''})\right| > \left|\Delta\alpha\left(\sigma^2|r = r'\right)\right|$. In this case, the positive term dominates the negative and $\Delta\alpha/\Delta\sigma^2 > 0$ (positive relationship). If, on the other hand, the matching curve is not sufficiently steep, then $\Delta\alpha/\Delta\sigma^2 < 0$ (negative relationship).
2. Suppose assumption A holds. The matching curve is everywhere positively sloped (positively assortative matching), i.e., $\Delta r/\Delta\sigma^2 > 0$. In this case, both terms in Eq. (15) are negative and $\alpha(m(\sigma^2), \sigma^2)$ is everywhere negatively sloped (negative relationship).
3. Suppose assumption C holds. The matching is simultaneously negatively and positively assortative. For all σ's that are less than $\hat{\sigma}^2(r_{n-1}, r_n)$ the matching is negatively assortative and the conclusions from case 1 hold. Also, for all σ's that are greater than $\hat{\sigma}^2(r_1, r_2)$ the matching is positively assortative and the conclusions from case 2 hold.

As we argued in Sect. 4.3, in any PAME, the bonus that an agent receives should be inversely related to his degree of risk aversion. This result has implications about equilibrium relationship between incentives and fixed salaries (fees) in any principal-agent market. Lafontaine (1992) observes that if the franchisors (principals) have all the bargaining power the participation constraint of the franchisee (agent) is binding and therefore royalties and franchise fees should be negatively correlated. However, as she puts it on pp.280–281, "*...no systematic negative or (positive) correlation was found between the two fees. ..*"

Nonetheless, participation constraints need not be binding even when we begin by assuming that principals have all the bargaining power (which is what we do in this paper). In fact, in a MPCE, only the participation constraint of the marginal agent is binding (i.e., $b = 0$). All other agents receive payments above their reservation utility (i.e., $b > 0$). Therefore, the relationship between royalties and franchise fees is not as clear cut as it was previously thought and it is not at all surprising to observe a non-negative correlation between the two.

To better illustrate the relationship between risk and the fixed fee $B = \beta + b$, differentiate Eq. (3) with respect to σ^2. This yields (using the Envelope Theorem),

$$\frac{\partial\beta}{\partial\sigma^2} = -\frac{\partial\alpha_{p,a}}{\partial\sigma^2} f(e_a(\alpha_{p,a})) - \frac{\partial\alpha_{p,a}}{\partial\sigma^2}\mu_p + \frac{\alpha_{p,a}^2 r_a}{2} > 0.$$

That is, as the variance increases the fixed salary increases, if the degree of risk aversion is fixed. But due to endogenous matching the degree of risk aversion cannot be assumed fixed. A change in r changes β as follows (using the Envelope Theorem),

$$\frac{\partial \beta}{\partial r} = -\frac{\partial \alpha_{p,a}}{\partial r} f(e_a(\alpha_{p,a})) - \frac{\partial \alpha_{p,a}}{\partial r}\mu_p + \frac{\alpha_{p,a}^2 \sigma_p^2}{2} > 0.$$

The total change in the fixed salary,

$$\Delta\beta = \beta(\sigma^{2''}, r'') - \beta(\sigma^{2'}, r'),$$

can be decomposed into two parts, the first being the change in β when the degree of risk aversion is kept unchanged,

$$\Delta\beta\left(\sigma^2|r = r'\right) = \beta(r', \sigma^{2''}) - \beta(r', \sigma^{2'}) = \int_{\sigma^{2'}}^{\sigma^{2''}} \frac{\partial \beta}{\partial \sigma^2} d\sigma^2$$

and the second being the change in β when the degree of risk aversion is allowed to move along the matching curve,

$$\Delta\beta(r|\sigma^2 = \sigma^{2''}) = \beta(\left[m(\sigma^{2'}) + \Delta r\right], \sigma^{2''}) - \beta(m(\sigma^{2'}), \sigma^{2''}) = \int_{r'}^{r''} \frac{\partial \beta}{\partial r} dr$$

Furthermore, an increase in σ^2 will lead to a change in the bonus b. Hence, the total effect of risk (rates of change) on the fixed fee $B = \beta + b$ is given by,

$$\frac{\Delta B}{\Delta\sigma^2} = \underset{(+)}{\frac{\Delta\beta\left(\sigma^2|r = r'\right)}{\Delta\sigma^2}} + \underset{(+,-)}{\frac{\Delta\beta(r|\sigma^2 = \sigma^{2''})}{\Delta\sigma^2}} + \underset{(+,-)}{\frac{\Delta b}{\Delta\sigma^2}} \gtreqless 0. \tag{16}$$

If the matching is negatively assortative, then an increase in σ^2 leads to a lower r. Hence the second term of Eq. (16) is negative and the third is positive (since a lower r commands a higher bonus in any PAME). If the matching curve is sufficiently flat, then the overall effect is positive, while if it is sufficiently steep the overall effect is negative. If the matching is positively assortative, then an increase in σ^2 leads to a higher r. The first two terms of Eq. (16) are positive, while the third is negative. If the matching curve is sufficiently steep then the overall effect is positive. It can be clearly seen that the equilibrium relationship between *royalties* and *fixed fees* could be positive, negative, or more importantly non-monotonic depending on the properties of the matching curve.

To illustrate the various properties of an equilibrium outcome in a principal-agent market we provide the following concrete examples. These examples also demonstrate how one can compute the equilibrium matching and the terms of the contracts.

6 Examples

A MPCE can be computed as follows. First, we calculate the $n \times n$ matrix Π using $\alpha_{p,a}$ and $\beta_{p,a}$ for all $(p, a) \in P \times A$ as defined in Sect. 3. Second, with the aid of Theorem 1 (for the first two examples) and the LP1 problem (for the third example) we find the assignment matrix x. Finally, we compute the minimum competitive bonus vector $\mathbf{b}_{\min}$, either by using the simple recursive algorithm of Sect. 4.2 (for the first two examples), or by solving the LP2 problem (for the third example).

The first example captures the positive part of the matching curve and depicts a negative relationship between risk and incentives. The second example yields a negatively sloped matching curve and a U-shaped relationship between risk and incentives. Finally, in the third example the matching is not globally assortative and the relationship between risk and incentives is non-monotonic. In all examples, we assume that $f(e) = e$, $C(e) = e^2/2$ and $\varepsilon_p \sim N(0, \sigma_p^2)$.

Example 1 (Positively assortative matching, PAM). Consider a market with four principals and four agents. The vector of the degrees of risk aversion is, $\mathbf{r} = (1, 2, 3, 4)$ and that of the asset variances is $\boldsymbol{\sigma}^2 = (2, 3, 4, 5)$ (see section 2 where these two vectors are first introduced). We can compute the expected profits [using Eq. (11)] of each principal-agent pair which are depicted below (agents are in columns and principals in rows),

(a, p)	1	2	3	4
4	0.055	0.038	0.029	0.024
3	0.071	0.05	0.038	0.031
2	0.1	0.071	0.055	0.045
1	0.166	0.125	0.1	0.083

For example, cell (4, 1) indicates that when agent 4 is paired with principal 1 the principal's expected profit is 0.055. This matrix is the matrix Π that we introduced in Sect. 4.1. Since, $1/\sqrt{r_1 \times r_2} = 1/\sqrt{1 \times 2} = .707 < \min \sigma^2 = 2$, assumption A holds and from Theorem 1 it follows that the matching curve must be everywhere positively sloped. So, the (permutation) matrix x which gives the optimal assignment in this problem is,

$$x = \begin{pmatrix} 0\,0\,0\,\mathbf{1} \\ 0\,0\,\mathbf{1}\,0 \\ 0\,\mathbf{1}\,0\,0 \\ \mathbf{1}\,0\,0\,0 \end{pmatrix}.$$

The minimum competitive bonus vector[19] and the level of incentives are,

(p, a)	(4, 4)	(3, 3)	(2, 2)	(1, 1)
$\mathbf{b}_{\min}$	0	0.007	0.024	0.078
$\alpha_{p,a}$	0.05	0.08	0.14	0.33

[19] This is in addition to the fixed salary β which has not been computed in these examples.

The demand sets (see Sect. 4.1 for the definition of the demand set) of each principal at $\mathbf{b}^*_{\min} = \{0.078, 0.024, 0.007, 0\}$ are,

$$\begin{aligned} D_1(\mathbf{b}^*_{\min}) &= \max_a \{0.088, 0.076, 0.064, 0.055\} = \{1\}, \\ D_2(\mathbf{b}^*_{\min}) &= \max_a \{0.047, 0.047, 0.043, 0.038\} = \{1, 2\}, \\ D_3(\mathbf{b}^*_{\min}) &= \max_a \{0.022, 0.031, 0.031, 0.029\} = \{2, 3\}, \\ D_4(\mathbf{b}^*_{\min}) &= \max_a \{0.005, 0.021, 0.024, 0.024\} = \{3, 4\}. \end{aligned}$$

Given the minimum competitive bonus vector, each principal is matched with an agent in his demand set. Since assumption A is satisfied the above outcome is also the unique PAME. Notice that the matching curve has a positive slope and incentives are negatively affected by risk. Moreover, all agents (except agent 4) receive positive rents.

Example 2 (Negatively assortative matching, NAM). Consider now the following 5×5 principal-agent market. Let $\mathbf{r} = (1, 2, 3, 4, 5)$ be the vector of the degrees of risk aversion and $\boldsymbol{\sigma}^2 = (.01, .05, .07, .09, .15)$ be the vector of the principals' asset variances. The matrix below depicts the profits which accrue to the principals from each matching.

(a, p)	1	2	3	4	5
5	0.4762	0.4000	0.3704	0.3448	0.2857
4	0.4808	0.4167	0.3906	0.3676	0.3125
3	0.4854	0.4348	0.4132	0.3937	0.3448
2	0.4902	0.4545	0.4386	0.4237	0.3846
1	0.4950	0.4762	0.4673	0.4587	0.4348

Following Theorem 1, the matching curve must be everywhere negatively sloped, since $1/\sqrt{r_4 \times r_5} = 1/\sqrt{4 \times 5} = 0.223 > \max \sigma^2 = 0.15$ (assumption B holds). The optimal matching is given by the (permutation) matrix x below,

$$x = \begin{pmatrix} \mathbf{1}\,0\,0\,0\,0 \\ 0\,\mathbf{1}\,0\,0\,0 \\ 0\,0\,\mathbf{1}\,0\,0 \\ 0\,0\,0\,\mathbf{1}\,0 \\ 0\,0\,0\,0\,\mathbf{1} \end{pmatrix}.$$

The minimum competitive bonus vector and the level of incentives are,

(p, a)	$(1, 5)$	$(2, 4)$	$(3, 3)$	$(4, 2)$	$(5, 1)$
$\mathbf{b}_{\min}$	0	0.0046	0.0227	0.0481	0.0831
$\alpha_{p,a}$	0.95	0.83	0.826	0.85	0.87

The equilibrium level of incentives (α) as a function of the variance are given in the table below,

σ^2	0.01	0.05	0.07	0.09	0.15
α	0.95	0.83	0.826	0.85	0.87

It can be seen from the above table that the data generated by this specification do not portray a monotonic relationship between risk and incentives. After a threshold level of variance incentives respond positively to an increase in risk. Overall $\alpha(\sigma^2)$ is U-shaped.

The demand sets of each principal at $\mathbf{b}^*_{\min} = \{0.0831, 0.0481, 0.0227, 0.0046, 0\}$ are,

$$D_1(\mathbf{b}^*_{\min}) = \max_a \{0.4119, 0.4421, 0.4267, 4{,}762, 0.4762\} = \{4, 5\},$$

$$D_2(\mathbf{b}^*_{\min}) = \max_a \{0.3931, 0.4064, 0.4121, 0.4121, 0.4000\} = \{3, 4\},$$

$$D_3(\mathbf{b}^*_{\min}) = \max_a \{0.3842, 0.3905, 0.3905, 0.3860, 0.3704\} = \{2, 3\},$$

$$D_4(\mathbf{b}^*_{\min}) = \max_a \{0.3756, 0.3756, 0.3710, 0.3623, 0.3448\} = \{1, 2\},$$

$$D_5(\mathbf{b}^*_{\min}) = \max_a \{0.3517, 0.3365, 0.3221, 0.3079, 0.2857\} = \{1\}.$$

As in Example 1, given the minimum competitive bonus vector, each principal is matched with an agent in his demand set. Since assumption B is satisfied the above outcome is also the unique PAME.

Example 3 (Not globally assortative matching, NGAM). Consider now the following 10×10 principal-agent market. Let $\mathbf{r} = (1, 2, 3, 4, 5, 6, 7, 8, 9, 10)$ be the vector of the degrees of risk aversion and $\boldsymbol{\sigma}^2 = (0.01, 0.05, 0.07, 0.12, 0.15, 0.20, 0.30, 0.75, 0.8, 1)$ be the vector of the variances. The matrix below depicts the profits which accrue to the principals from each matching.

(a, p)	1	2	3	4	5	6	7	8	9	10
10	0.4545	0.3333	0.2941	0.2273	0.2000	0.1667	0.125	0.05882	0.05556	0.04545
9	0.4587	0.3448	0.3067	0.2404	0.2128	0.1786	0.1351	0.06452	0.06098	0.05000
8	0.4630	0.3571	0.3205	0.2551	0.2273	0.1923	0.1471	0.07143	0.06757	0.05556
7	0.4673	0.3704	0.3356	0.2717	0.2439	0.2083	0.1613	0.08000	0.07576	0.06250
6	0.4717	0.3846	0.3521	0.2907	0.2632	0.2273	0.1786	0.09091	0.08621	0.07143
5	0.4762	0.4000	0.3704	0.3125	0.2857	0.2500	0.2000	0.1053	0.1000	0.08333
4	0.4808	0.4167	0.3906	0.3378	0.3125	0.2778	0.2273	0.1250	0.1190	0.1000
3	0.4854	0.4348	0.4132	0.3676	0.3448	0.3125	0.2632	0.1538	0.1471	0.1250
2	0.4902	0.4545	0.4386	0.4032	0.3846	0.3571	0.3125	0.2000	0.1923	0.1667
1	0.4950	0.4762	0.4673	0.4464	0.4348	0.4167	0.3846	0.2857	0.2778	0.2500

The optimal matching is given by the (permutation) matrix x below,[20]

$$x = \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 \end{pmatrix}$$

The optimal principal-agent matching is not globally assortative. For relatively low levels of risk, it is negatively assortative, while for relatively high values of risk it is positively assortative.[21] The competitive bonus vector (which was obtained by solving the LP2 problem, using the Maple software) and the level of incentives are,

(p, a)	(1, 10)	(10, 9)	(9, 8)	(8, 7)	(2, 6)	(3, 5)	(4, 4)	(5, 3)	(6, 2)	(7, 1)
$\mathbf{b}_{\min}$	0	0.0042	0.00976	0.01795	0.03105	0.04935	0.07465	0.10695	0.1515	0.22365
$\alpha_{p,a}$	0.91	0.1	0.135	0.16	0.77	0.74	0.68	0.69	0.71	0.77

The relationship between risk and incentives is summarized in the table below,

σ^2	0.01	0.05	0.07	0.12	0.15	0.20	0.30	0.75	0.80	1
α	0.91	0.77	0.74	0.68	0.69	0.71	0.77	0.16	0.135	0.1

Notice that incentives exhibit a sideways-S relationship with risk.[22]

7 Concluding remarks

We employ the assignment game of Shapley and Shubik (1972) to study the endogenous matching patterns in a market that consists of heterogenous principals and agents. We show that, in general, the equilibrium matching is non-assortative. We then characterize the equilibrium relationship between risk and performance pay and risk and fixed salary. This is the first paper that characterizes the equilibrium matching, to its fullest possible extent, based on the Holmstrom and Milgrom (1987) principal-agent model. This model has been used extensively in the empirical literature and therefore

[20] The optimal assignment was derived by solving the LP1 problem using the Maple software.

[21] This example confirms the predictions of Theorem 1. Based on that Theorem, for $\sigma^2 \leq \hat{\sigma}^2(r_{n-1}, r_n) = 1/\sqrt{9 \times 10} = 0.105$ the matching curve has a negative slope, while for $\sigma^2 \geq \hat{\sigma}^2(r_1, r_2) = 1/\sqrt{1 \times 2} = 0.707$ it has a positive slope. As we have already alluded to, if there are σ^2's in $\left[\hat{\sigma}^2(r_{n-1}, r_n), \hat{\sigma}^2(r_1, r_2)\right]$, as it is the case in this example, we cannot utilize Theorem 1 to completely characterize the shape of the matching curve.

[22] Due to space limitations we do not present the demand sets $D_p(\mathbf{b}^*_{\min})$, $p \in P$ for this example.

we hope that our results will be of value to empirical researchers who wish to study a principal-agent market.

A Appendix: Proof of Lemma 1

Suppose by way of contradiction that this is not the case, that is, $(\mathcal{C}, m)$ is a PAME but the assignment m is not optimal. Therefore, there exist at least two pairs (p, a) and $(p^{'}, a^{'})$ such that $m(p) = a$ and $m(p') = a'$ but nevertheless,

$$\Pi_{p,a} + \Pi_{p^{'},a^{'}} < \Pi_{p,a^{'}} + \Pi_{p^{'},a}. \tag{A1}$$

We assume that agent $a^{'}$ has a lower degree of risk aversion, i.e., $r_a > r_{a^{'}}$. Thus, (A1) can be rewritten as,

$$\Pi_{p,a^{'}} - \Pi_{p,a} > \Pi_{p^{'},a^{'}} - \Pi_{p^{'},a}. \tag{A2}$$

Let $b_{p,a}$ and $b_{p^{'},a^{'}}$ be the bonuses that the two agents receive, respectively. Since $(\mathcal{C}, m)$ is a PAME, it must be the case that,

$$b_{p^{'},a^{'}} \geq \left[\Pi_{p,a^{'}} - \left(\Pi_{p,a} - b_{p,a}\right)\right], \tag{A3}$$

otherwise the pair $(p, a^{'})$ would block the initial matching and both principal p and agent $a^{'}$ would become better off. But this implies that,

$$\left(b_{p^{'},a^{'}} - b_{p,a}\right) \geq \left(\Pi_{p,a^{'}} - \Pi_{p,a}\right) > \left(\Pi_{p^{'},a^{'}} - \Pi_{p^{'},a}\right) \Longrightarrow$$
$$\left(\Pi_{p^{'},a} - b_{p,a}\right) > \left(\Pi_{p^{'},a^{'}} - b_{p^{'},a^{'}}\right),$$

using (A2) and (A3). Principal $p^{'}$ is better off blocking the current assignment and getting matched with agent a. In this case his reduction in profits (since he moved to the higher degree of risk aversion agent) is less than the decrease in the bonus, which leaves room for mutual benefit. Hence $(\mathcal{C}, m)$ is not a PAME. Contradiction. The proof is similar if $r_a < r_{a^{'}}$. □

B Appendix: Proof of Lemma 2

(1) Fix two agents j and k in A with $r_j < r_k$. It follows that,[23]

$$M^{(j-k)}(\sigma_p^2) = \Pi_{p,j}(\sigma_p^2) - \Pi_{p,k}(\sigma_p^2)$$

[23] Based on our assumptions, all functions are continuously differentiable by an appeal to the Implicit Function Theorem.

$$= \int_{r_k}^{r_j} \frac{\partial \Pi_p(\sigma_p^2)}{\partial r_a} dr_a$$

$$= -\int_{r_j}^{r_k} \frac{\partial \Pi_p(\sigma_p^2)}{\partial r_a} dr_a$$

$$= \int_{r_j}^{r_k} \frac{\left[\alpha_{p,a}(r_a, \sigma_p^2)\right]^2 \sigma_p^2}{2} dr_a \geq 0,$$

where the first equality follows from the definition of $M^{(j-k)}(\sigma_p^2)$, the second and third equalities are obvious and the last inequality follows from Eq. (9).
(2) Suppose that r_j increases to $r_{j'}$.

$$\begin{aligned} M^{(j-k)}(\sigma_p^2) - M^{(j'-k)}(\sigma_p^2) &= \Pi_{p,j}(\sigma_p^2) - \Pi_{p,k}(\sigma_p^2) - \left(\Pi_{p,j'}(\sigma_p^2) - \Pi_{p,k}(\sigma_p^2)\right) \\ &= \Pi_{p,j}(\sigma_p^2) - \Pi_{p,j'}(\sigma_p^2) \\ &= \int_{r_j}^{r_{j'}} \frac{\left[\alpha_{p,a}(r_a, \sigma_p^2)\right]^2 \sigma_p^2}{2} dr_a \geq 0. \end{aligned}$$

The proof is similar when r_k decreases to $r_{k'}$.
(3) Since $\alpha_{p,a}$ is bounded, i.e., $\left[\alpha_{p,a}(r_a, \sigma_p^2)\right]^2 \in [0, 1]$, it is clear that $M_p^{(j-k)}(0) = 0$.
(4)

$$\begin{aligned} \frac{\partial M^{(j-k)}(\sigma_p^2)}{\partial \sigma_p^2} &= \frac{\partial \left(\int_{r_j}^{r_k} \frac{\left[\alpha_{p,a}(r_a,\sigma_p^2)\right]^2 \sigma_p^2}{2} dr_a \right)}{\partial \sigma_p^2} \\ &= \int_{r_j}^{r_k} \frac{\partial \left(\frac{\left[\alpha_{p,a}(r_a,\sigma_p^2)\right]^2 \sigma_p^2}{2} \right)}{\partial \sigma_p^2} dr_a \\ &= \int_{r_j}^{r_k} \left(\alpha_{p,a}(r_a, \sigma_p^2)\sigma_p^2 \frac{\partial \alpha_{p,a}(r_a, \sigma_p^2)}{\partial \sigma_p^2} + \frac{\left[\alpha_{p,a}(r_a, \sigma_p^2)\right]^2}{2} \right) dr_a \\ &= \int_{r_j}^{r_k} \left[\alpha_{p,a}(r_a, \sigma_p^2)\right]^2 \left[\frac{1}{2} + \frac{r_a}{A/\sigma_p^2 - r_a} \right] dr_a, \end{aligned} \tag{A4}$$

where the last equality was derived by using Eq. (7) and,

$$A = -f^{'}(e(\alpha_{p,a}))\frac{de_a}{d\alpha_{p,a}} + (1-\alpha_{p,a})f^{''}(e(\alpha_{p,a}))\left(\frac{de_a}{d\alpha_{p,a}}\right)^2 + (1-\alpha_{p,a})f^{'}(e(\alpha_{p,a}))\frac{d^2e_a}{d\alpha_{p,a}^2} < 0,$$

[from the first three terms of Eq. (6), which are independent of σ_p^2 and r_a]. Fix the limits of integration r_j and r_k. The integrand in (A4) is positive for all $r \in [r_j, r_k]$ if $\sigma_p^2 < -A/r_k$ and negative for all $r \in [r_j, r_k]$ if $\sigma_p^2 > -A/r_j$. Hence, the integral (since $\alpha_{p,a}^2 \in [0, 1]$) is positive for low σ^2's and negative for high σ^2's. It follows by continuity that there exists a $\hat{\sigma}^2(r_j, r_k) \in [-A/r_k, -A/r_j]$ such that the integral is zero, i.e., $\partial M^{(j-k)}(\hat{\sigma}^2)/\partial\sigma_p^2 = 0$.

We will now show that: i) $\partial M^{(j-k)}(\sigma_p^2)/\partial\sigma_p^2 > 0$ for all $\sigma_p^2 < \hat{\sigma}^2(r_j, r_k)$ and ii) $\partial M^{(j-k)}(\sigma_p^2)/\partial\sigma_p^2 < 0$, for all $\sigma_p^2 > \hat{\sigma}^2(r_j, r_k)$. This is equivalent to showing that $\hat{\sigma}^2(r_j, r_k)$ is unique. First, we know already that for all $\sigma_p^2 < -A/r_j$, $\partial M^{(j-k)}(\sigma_p^2)/\partial\sigma_p^2 > 0$ and for all $\sigma_p^2 > -A/r_k$, $\partial M^{(j-k)}(\sigma_p^2)/\partial\sigma_p^2 < 0$. But $\hat{\sigma}^2(r_j, r_k) \in [-A/r_k, -A/r_j]$ and therefore the proof is not complete yet. To complete it, we differentiate Eq. (A4) with respect to σ^2. This yields,

$$\begin{aligned}\frac{\partial^2 M^{(j-k)}(\sigma_p^2)}{\partial\left(\sigma_p^2\right)^2} &= \int_{r_j}^{r_k}\left[2\left[\alpha_{p,a}(r_a,\sigma_p^2)\right]\frac{\partial\alpha_{p,a}(r_a,\sigma_p^2)}{\partial\sigma_p^2}\left[\frac{1}{2}+\frac{r_a\sigma_p^2}{A-r_a\sigma_p^2}\right]\right.\\&\qquad\left.+\frac{\left[\alpha_{p,a}(r_a,\sigma_p^2)\right]^2 r_a A}{\left(A-r_a\sigma_p^2\right)^2}\right]dr_a\\&= \int_{r_j}^{r_k}\left[\frac{\left[\alpha_{p,a}(r_a,\sigma_p^2)\right]^2 r_a\left(2A+r_a\sigma_p^2\right)}{\left(A-r_a\sigma_p^2\right)^2}\right]dr_a \qquad \text{(A5)}\end{aligned}$$

using Eq. (7). Equation (A5) changes sign once, i.e., it is negative for low σ^2's and positive for high σ^2's (recall $A < 0$). Now suppose by way of contradiction that $\hat{\sigma}^2(r_j, r_k)$ is not unique. Since Eq. (A4) is continuously differentiable, positive for low σ^2's and negative for high σ^2's, it must be the case that the number of $\hat{\sigma}^2(r_j, r_k)$'s is odd. Suppose that the number of $\hat{\sigma}^2(r_j, r_k)$'s is three (a similar argument can be used when the number is more than three). We proved above that $M^{(j-k)}(\sigma_p^2)$ is initially increasing and eventually decreasing. This implies that $M^{(j-k)}(\sigma_p^2)$ must have at least one maximum point. There are two cases that we have to consider.

Case 1 The $M^{(j-k)}(\sigma_p^2)$ function has two local maxima and one minimum. This implies that the function must be locally concave (when the first local maximum is attained), then locally convex when the minimum is reached and finally locally concave again at the second local maximum. This implies that the second derivative of $M^{(j-k)}(\sigma_p^2)$ must change sign at least twice. Contradiction to the fact that Eq. (A5) changes sign once.

Case 2 The $M^{(j-k)}(\sigma_p^2)$ function has one maximum and two inflection points. By the definition of an inflection point, the second derivative of $M^{(j-k)}(\sigma_p^2)$ must change sign at least twice. Contradiction to the fact that Eq. (A5) changes sign once.

Thus, $\partial M^{(j-k)}(\sigma_p^2)/\partial\sigma_p^2 > 0$ for $\sigma_p^2 < \hat{\sigma}^2(r_j, r_k)$, $\partial M^{(j-k)}(\sigma_p^2)/\partial\sigma_p^2 < 0$ for $\sigma_p^2 > \hat{\sigma}^2(r_j, r_k)$ and $\partial M^{(j-k)}(\sigma_p^2)/\partial\sigma_p^2 = 0$ at $\sigma_p^2 = \hat{\sigma}^2(r_j, r_k)$.
(5) Fix the limits of integration $\left[r_j, r_k\right]$ and find the $\hat{\sigma}^2$ such that $\partial M^{(j-k)}(\hat{\sigma}^2)/\partial\sigma_p^2 = 0$. This induces an $r^* \in \left[r_j, r_k\right]$ such that the integrand in (A4) is positive for $r \leq r^*$ and negative for $r \geq r^*$. Moreover, the positive area equals the negative one. Now assume the lower limit changes to $r_j^{'}$, where $r_j^{'} < r_j$. This will increase the positive part of the integrand and consequently $\partial M^{(j^{'}-k)}(\hat{\sigma}^2)/\partial\sigma_p^2 > 0$. As we proved above, the $\hat{\sigma}^2$ is unique and therefore $\hat{\sigma}^2(r_j^{'}, r_k) > \hat{\sigma}^2(r_j, r_k)$. Now suppose that the upper limit changes to $r_k^{'}$, where $r_k^{'} < r_k$. The decrease of the upper limit reduces the negative part and therefore $\partial M^{(j-k^{'})}(\hat{\sigma}^2)/\partial\sigma_p^2 > 0$, which again implies that $\hat{\sigma}^2(r_j, r_{k'}) > \hat{\sigma}^2(r_j, r_k)$. □

C Appendix: Proof of Proposition 1

Based on the definition of MPCE (see Sect. 4.1), we must show that: $i)$ $a \in D_p(\mathbf{b}_{\min}^*)$ for every $(p, a) \in A \times P$ such that $m^*(p) = a$, and $ii)$ there is no other equilibrium bonus vector $\mathbf{b}^{'} \leq \mathbf{b}_{\min}^*$.

We begin by assuming that assumption A holds. A similar argument will work under assumption B. Recall that under assumption A, $m(1) = 1$, $m(2) = 2, \ldots$ and $m(n) = n$ (positively assortative matching). In our $n \times n$ market it is easy to see that in any optimal assignment every principal is matched with one agent and vice versa. To prove $i)$ consider principal l who is matched with agent l. We need to show that,

$$\Pi_{l,l} - b_{l,\min}^* \geq \Pi_{l,k} - b_{k,\min}^*,$$

for all $(k, k) \in P \times A$, with $m^*(k) = k$. First consider $k > l$. So the above inequality can be re-written as,

$$\Pi_{l,l} - \Pi_{l,k} \geq b_{l,\min}^* - b_{k,\min}^*.$$

Further assume that there are h principal-agent pairs between the pairs (l, l) and (k, k) . Using the simple algorithm of section 4.2 the above inequality becomes,

$$\Pi_{l,l} - \Pi_{l,k} \geq b_{l,\min}^* - b_{k,\min}^*$$
$$= \underbrace{\left(\Pi_{l+1,l} - \Pi_{l+1,l+1}\right) + \left(\Pi_{l+2,l+1} - \Pi_{l+2,l+2}\right) + \cdots + \left(\Pi_{k-1,l+h-1} - \Pi_{k-1,l+h}\right) + \left(\Pi_{k,l+h} - \Pi_{k,k}\right)}_{(h+1)\text{ terms}} \Leftrightarrow$$

$$\begin{aligned}&\left(\Pi_{l,l}-\Pi_{l,l+1}\right)+\left(\Pi_{l,l+1}-\Pi_{l,l+2}\right)+\cdots+\left(\Pi_{l,l+h-1}-\Pi_{l,l+h}\right)+\left(\Pi_{l,l+h}-\Pi_{l,k}\right)\\&\quad\geq\left(\Pi_{l+1,l}-\Pi_{l+1,l+1}\right)+\left(\Pi_{l+2,l+1}-\Pi_{l+2,l+2}\right)\\&\qquad+\cdots+\left(\Pi_{k-1,l+h-1}-\Pi_{k-1,l+h}\right)+\left(\Pi_{k,l+h}-\Pi_{k,k}\right).\end{aligned}$$

Now note that the above inequality is satisfied because each term on the left hand side is greater than its counterpart on the right hand side. This is because, under assumption A, all $\left(r,\sigma^2\right)$ pairs are located on the decreasing part of the $M^{(\cdot,\cdot)}(\cdot)$ functions, which implies that $\left(\Pi_{l,l}-\Pi_{l,l+1}\right)\geq\left(\Pi_{l+1,l}-\Pi_{l+1,l+1}\right)$, $\left(\Pi_{l,l+1}-\Pi_{l,l+2}\right)\geq\left(\Pi_{l+2,l+1}-\Pi_{l+2,l+2}\right)$ and so on. The proof is similar when $k<l$.

Finally, it is easy to see that $\mathbf{b}^*_{\min}$ is the minimum equilibrium bonus vector, since each principal pays his assigned agent the minimum payment necessary so that no other principal among the ones who are employing more risk averse agents finds it profitable to hire him. □

D Appendix: Proof of Proposition 2

Suppose that assumption A holds (the proof is similar when assumption B holds). Recall that under assumption A, $m(1)=1$, $m(2)=2,\ldots,$ and $m(n)=n$ (positively assortative matching). For this reason in the remaining of this proof $p=a$ when $m(p)=a$. A consequence of the simple algorithm which computes $\mathbf{b}_{\min}$ (Sect. 4.3) is that the induced demand sets have a special structure (see Example 1 in section 6 for a concrete illustration). In particular, $D_p(\mathbf{b}_{\min})=\{m(p),m(p-1)\}$ for all p. In other words,

$$\Pi_{p,m(p)}-b_{m(p),\min}=\Pi_{p,m(p-1)}-b_{m(p-1),\min}>\Pi_{p,m(p')}-b_{m(p'),\min},\quad\text{for all } p,\,p'.$$

Suppose by way of contradiction that (C,m) is a PAME but not a MPCE. In particular, let $(\mathbf{u},\mathbf{b})$ be a PAME, such that $\mathbf{b}\geq\mathbf{b}_{\min}$ with $\mathbf{b}\neq\mathbf{b}_{\min}$. Also let m be the induced optimal assignment and $m(p')=a'$ be the first pair (starting from the lowest b_a and going up) for which $b_{a'}>b_{a',\min}$. Also recall that assumption A implies,

$$M^{(a'-a'')}(\sigma_p^2)=\Pi_{p,a'}\left(\sigma_p^2\right)-\Pi_{p,a''}\left(\sigma_p^2\right)$$

is strictly increasing in σ_p^2 for any a', a'' with $r_{a'}<r_{a''}$. This fact is used extensively in this proof.

First observe that principal p' has room to lower $b_{a'}$ without him loosing his assigned agent to principal $p'+1$ (who before the increase in $b_{a',\min}$ was indifferent between agent a' and $a'+1$). Because of the special structure of the demand sets, the above assertion holds for any $p''>p'$ (and not just for $p'+1$). Hence, for principal p' not to be able to benefit by lowering his bonus it must be that by doing so he looses his agent to principal $p'-1$, i.e.,

$$\Pi_{p'-1,a'-1}-b_{a'-1}=\Pi_{p'-1,a'}-b_{a'}.\tag{A6}$$

The above equality cannot hold for any principal p with $p < p' - 1$. If that was the case, then

$$\Pi_{p'-1,a'-1} - b_{a'-1} < \Pi_{p'-1,a'} - b_{a'},$$

contradicting the fact that $m(p'-1) = a'-1$. The above inequality follows from the assumption that the $M^{(\cdot,\cdot)}(\cdot)$ functions are strictly increasing. Furthermore, if (A6) holds then,

$$\Pi_{p'',a''} - b_{a''} > \Pi_{p'',a'} - b_{a'} \tag{A7}$$

for all $p'' < p' - 1$ and $m(p'') = a''$. This is because,

$$M^{(a'-1,a')}\left(\sigma^2_{p''}\right) > M^{(a'-1,a')}\left(\sigma^2_{p'-1}\right), \quad \text{for all } p'' < p' - 1,$$

which implies,

$$\Pi_{p'',a'-1} - b_{a'-1} > \Pi_{p'',a'} - b_{a'}. \tag{A8}$$

But since $m(p'') = a''$ it must be the case that (A7) holds, for if (A7) held with equality then (A7) and (A8) would imply,

$$\Pi_{p'',a''} - b_{a''} = \Pi_{p'',a'} - b_{a'} < \Pi_{p'',a'-1} - b_{a'-1}$$

contradicting $m(p'') = a''$. So if principal $p'-1$ lowers his bonus slightly no principal $p'' < p'-1$ will find it in his best interest to break the current assignment and employ $a'-1$. Now let's turn to principals $p'' \geq p'$.

For principal p', it must be,

$$\Pi_{p',a'} - b_{a'} \geq \Pi_{p',a'-1} - b_{a'-1}, \tag{A9}$$

since $m(p') = a'$. If (A9) is satisfied with strict inequality then (following a similar argument to the one that justified (A7)),

$$\Pi_{p'',a''} - b_{a''} > \Pi_{p'',a'} - b_{a'} \tag{A10}$$

for all $p'' > p'$ and $m(p'') = a''$. (A7), (A9, with equality) and (A10) imply that principal $p'-1$ will become better off by lowering his bonus, contradicting the fact that (C, m) is a PAME. Hence the only possibility is for (A9) to be satisfied with equality. But this together with (A6) contradict assumption A. $\square$

References

Ackerberg DA, Botticini M (2002) Endogenous matching and the empirical determinants of contract form. J Polit Econ 110:564–592

Anderson A, Smith L (2003) Assortative matching and reputation, working paper

Becker G (1973) A theory of marriage: part I. J Polit Econ 81:813–846

Besley T, Ghatak M (2005) Competition and Incentives with Motivated Agents. American Economic Review 95:616–636

Crawford V, Knoer EM (1981) Job matching with heterogeneous firms and workers. Econometrica 49:437–450

Dam K, Perez-Castrillo D (2006) The principal-agent matching market, frontiers in economic theory. Berkeley Electronics, Frontiers of theoretical Economics, 2, N.1
Demange G, Gale D, Sotomayor M (1986) Multi-item auctions. J Polit Econ 94:863–872
Gale D, Shapley L (1962) College admissions and the stability of marriage. Am Math Month 69:9–15
Gretsky NE, Ostroy JM, Zame WR (1992) The nonatomic assignment model. Economic Theory 2:103–127
Holmstrom B, Milgrom P (1987) Aggregation and linearity in the provision of intertemporal incentives. Econometrica 55:303–328
Lafontaine F (1992) Agency theory and franchising: some empirical results. RAND J Econ 23:263–283
Legros P, Newman AF (2002) Monotone matching in perfect and imperfect worlds. Rev Econ Stud 69:925–942
Leonard HB (1983) Elicitation of honest preferences for the assignment of individuals to positions. J Polit Econ 91:461–479
Prendergast C (2002) The tenuous trade-off between risk and incentives. J Polit Econ 110:1071–1102
Roth A, Sotomayor M (1990) Two sided matching: a study in game-theoretic modeling and analysis. Econ Soc Monogr No. 18. Cambridge University Press, New York
Serfes K (2005) Risk sharing vs. incentives: contract design under two-sided heterogeneity. Econ Lett 88:343–349
Shapley LS, Shubik M (1972) The assignment game I: The core. Int J Game Theory 1:111–130
Shimer R, Smith L (2000) Assortative matching and search. Econometrica 68:343–370
Wright DJ (2004) The risk and incentives trade-off in the presence of heterogeneous managers. J Econ 83:209–223