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A LOCATION MODEL WITH PREFERENCE FOR VARIETY*

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We propose a new location model where consumers are allowed to make multiple purchases (i.e., one unit from each firm). This model may fit many markets (e.g. newspapers, credit cards) better than existing models. A common feature of these markets is that some consumers are loyal to one brand, while others consume more than one product. Our model yields predictions consistent with this observation. If firms are allowed to choose their locations on the interval, then spatial differentiation may not be maximal and in some cases it may even be minimal. Thus, under certain conditions, we restore Hotelling's Principle of Minimum Differentiation.

I. INTRODUCTION

HOTELLING (1929), IN HIS SEMINAL PAPER, INTRODUCED A MODEL OF SPATIAL COMPETITION where two firms are located at the two end points of a busy street producing homogeneous products. Consumers have unit demands, are uniformly distributed on the street and each one buys from the firm which offers the best deal in terms of price and distance that has to be travelled. Later versions of that model have relaxed a number of Hotelling's assumptions, but the one that is firmly maintained is that consumers purchase from one firm exclusively. Implicitly, this says that consumers do not care for diversity. This may be true under the initial hypothesis that the two products are homogenous, but not necessarily under the broader interpretation of the model by which the products are differentiated and the distance that a consumer has to travel serves as a measure of disutility. In many markets, consumers need not purchase more than one unit of a given brand but, nevertheless, variety is valued. For example, it does not make

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sense to buy two copies of the same newspaper but, for *some* people, it makes perfect sense to read two different newspapers. Gentzkow [2003] studies the newspapers market in Washington D.C. and finds that one third of the consumers in his sample read more than one newspaper. A second example can be drawn from the credit cards market. People usually hold only one credit card from a given issuer, but a second card, from a different issuer, provides the holder with a higher credit limit and consequently a higher utility. Furthermore, there is a great deal of horizontal preference heterogeneity regarding the typesetting software ScientificWord (or WorkPlace) and Microsoft Word. Nevertheless, utility increases if both are used, since this enhances a person's ability to communicate and collaborate with others. Finally, consider two economics journals differentiated by the mix of applied theory and applied econometrics papers they publish. Although users' strength of preferences for these two products varies, one thing seems to be common among all scholars: appreciation of variety (see McCabe [2002]). Our aim in this paper is to develop a modeling framework capable of capturing some of the key features of the above examples.[1]

We extend Hotelling's model by allowing consumers to make multiple purchases (i.e., to buy one unit from each firm). The model we propose generates a very realistic equilibrium where some consumers remain loyal to one brand, while another group of consumers buys both brands. We begin by assuming that the firms' horizontal locations are fixed at the two extreme points of the unit interval. Our analysis yields several interesting new insights. We show that when the incremental utility from buying a second brand is below a threshold, then the equilibrium is the same as the one in the standard Hotelling model. That is, half of the consumers buy from one firm and half from the other exclusively. When the incremental utility exceeds that threshold, a new equilibrium emerges which is characterized by a fraction of consumers who purchase both brands. Overall, this says that the desire for diversity on the part of the consumers has to be sufficiently strong for the firms to switch to an equilibrium where some consumers find it beneficial to buy both brands. The presence of a group of consumers which uses both products acts as a buffer which alters the nature of price competition significantly. When a firm lowers its price, the demand for its product increases, but not necessarily at the expense of its rival. This produces interesting comparative statics. For instance, as products become *more* differentiated (i.e., transportation cost increases), equilibrium profits *decrease* when in equilibrium some consumers consume both brands. The

[1] There is a growing interest for empirical applications based on models which combine spatial competition with preference for diversity, or for multiple purchases discrete choice models. For instance, Pinske *et al.* [2002] propose a spatial model to study the nature of competition in the U.S. wholesale gasoline markets. Hendel [1999] introduces a multiple purchases discrete choice model to study the demand for personal computers.

non-cooperative outcome is, for a certain range of parameter values, inefficient. In particular, output is lower than the socially optimal level. This contrasts with the standard Hotelling model, which always yields an efficient equilibrium for fixed firm locations and a covered market.[2]

Next, we endogenize the locations of the firms. Hotelling claimed that firms will agglomerate in the middle of the unit interval (*Principle of Minimum Differentiation*). d'Aspermont *et al.* [1979] showed that this claim is incorrect. The reason is that when firms are sufficiently close to each other, a pure strategy price equilibrium fails to exist. They also demonstrated that if the linear transportation cost is replaced by a quadratic one, then the price equilibrium is restored, but firms locate at the extremes, rather than the middle (*Principle of Maximum Differentiation*).[3] In the present paper, we assume a quadratic transportation cost and we show that if the incremental utility from consuming a second product exceeds a threshold, then spatial differentiation will be *minimal*.[4] The driving force behind the firms' tendency to move closer to each other is an *aggregate demand creation* effect. This effect is not present in single purchase models. Thus, this model is capable of restoring Hotelling's *Principle of Minimum Differentiation*. Furthermore, under a different set of conditions, spatial differentiation may be *maximal* or *intermediate*.

Consumers in our model have an elastic demand of a special kind. Although they cannot buy more than one unit from the same firm, they are allowed to purchase up to two units, each one from a different firm. A number of papers in the literature (e.g. Anderson *et al.* [1989], Anderson and Neven [1991], Hamilton *et al.* [1994] and Rath and Zhao [2001]) have already introduced models with an elastic demand. In those papers, however, consumers are allowed to buy more than one unit from the same firm, but cannot purchase from other firms (and hence the issue of variety does not arise). The results with regards to product selection are also different from ours. Hamilton *et al.* assume a linear transportation cost paid for every unit

[2] Hagiu [2004] develops a model where consumers, as in our model, value variety. On a Salop circle he allows consumers to buy more than one brand. Assuming linear transportation costs, he shows that, in general, a symmetric pure strategy equilibrium does not exist. In particular, this happens when the number of varieties demanded by consumers is strictly greater than two.

[3] Other remedies (all in the context of a purchase from a *single* firm) to the problem include: mixed strategy equilibrium, (e.g., Gal-Or [1982]), heterogeneity in consumers' tastes, (e.g., de Palma *et al.* [1985]), different utility functions (e.g. Economides [1986]), Cournot-type competition (e.g. Anderson and Neven [1991]) and a non-uniform distribution (e.g., Tabuchi and Thisse [1995]). Gal-Or does not restore the Principle of Minimum Differentiation, while de Palma *et al.* do. Economides shows that differentiation will never be minimal. However, not all equilibria exhibit maximal product differentiation. Anderson and Neven show that competition between Cournot-type oligopolists who discriminate over space leads to spatial agglomeration. Tabuchi and Thisse study a model with a non-uniform distribution and show that differentiation may not be maximal.

[4] Minimal differentiation continues to hold even when the quadratic cost is replaced by a linear one. A price equilibrium exists for *all* location configurations, provided that the incremental utility exceeds a certain threshold.

of the product and they show that a price equilibrium, in pure strategies, may not exist. Rath and Zhao assume a quadratic lump-sum transportation cost and show that firms may locate at the extremes or strictly inside the interval (including the middle point) depending upon the ratio of the reservation price and the transportation cost parameter.[5]

Spatial models have been utilized extensively in the literature to address a number of interesting questions related to price discrimination, entry decisions, product variety, vertical integration and market foreclosure, to mention a few. The modeling framework presented in this paper may fit many markets better than the existing models and it can be adapted to study old and new issues through different lenses. For example, a large body of the spatial price discrimination literature (e.g., Fudenberg and Tirole [2000], Liu and Serfes [2004] and Shaffer and Zhang [2000]), builds on the presumption that firms can easily segment consumers into two groups: own customers and rival firms' customers. If consumers buy only from one firm, then this distinction is clear, but not when some consumers purchase from both firms. As a second example, consider the incentives of vertically integrated firms in the market for broadband access (e.g., a content provider (upstream firm) and an Internet service provider (downstream firm)) to practice conduit and/or content discrimination, (see Rubinfeld and Singer [2001]). The major element of differentiation comes from the various contents that a service provider carries. In this case, it seems natural to assume that consumers have heterogeneous preferences for different contents, e.g. music, video games, movies, news etc. At the same time, though, variety is valued. A standard location model would ignore the preference bias towards variety. On the other hand, a representative consumer model (e.g. Singh and Vives [1984]), where preferences are symmetric, would miss the key element of preference heterogeneity.[6]

The rest of the paper is organized as follows. Section 2 presents the model. In section 3, we solve for the Nash equilibrium, assuming that the locations of the firms are fixed at the two endpoints of the unit interval, and we compare the non-cooperative outcome with the social optimum. In section 4, we endogenize product selection by allowing the firms to choose their horizontal locations. We conclude in section 5. The proof of proposition 2 can be found in the appendix.

[5] Caplin and Nalebuff [1991] introduce a general model with multi-dimensionally differentiated products and prove existence of pure strategy price equilibrium, by showing that the profit functions are quasi-concave in a firm's own price (proposition 4, p.39). Their model encompasses many of the alternative approaches to the theory of differentiated products (e.g., C.E.S. preferences, characteristics approach models and multi-dimensional probabilistic choice models) as special cases. Nevertheless, our framework does not satisfy the assumptions made by Caplin and Nalebuff and therefore it is not nested in theirs. As we demonstrate later, our model yields profit functions which are not quasi-concave.

[6] In Kim and Serfes [2004], we build on the present model to investigate the incentives of two vertically integrated firms to engage in content and/or network discrimination.

II. THE DESCRIPTION OF THE MODEL

There are two firms, A and B, who produce differentiated brands and are located at the two endpoints of the unit interval [0, 1]. A unit mass of consumers is uniformly distributed on the [0, 1] interval. Each consumer can buy at most one unit from a given firm and has the following valuation: $V = \alpha q_i + \theta q_A q_B$, where $i = A,B$ and q_i represents the quantity of brand i that a consumer buys, with $q_i \in \{0, 1\}$. These valuations do not depend on a consumer's particular horizontal location. Hence, if a consumer buys only one brand, his valuation is equal to $V(1) = \alpha > 0$, while if he buys both brands, his valuation is $V(2) = \alpha + \theta$, with $\theta \geqslant 0$. Moreover, we assume diminishing incremental (marginal) utility, i.e, $\theta \leqslant \alpha$. Thus $\theta = V(2) - V(1)$ denotes the incremental utility from purchasing a second product, with $\theta \in [0, V(1)]$. In addition, consumers incur a disutility from not being able to purchase their 'ideal' brand. A consumer who is located at $x \in [0, 1]$ incurs a disutility equal to tx^2 if he buys brand A, a disutility equal to $t(1 - x)^2$ if he buys brand B and a disutility equal to $tx^2 + t(1 - x)^2$ if he buys both, with $t > 0$.

If a consumer, who is located at $x \in \left[0, \frac{1}{2}\right]$, buys from both firms, then the transportation cost he incurs is $tx^2 + t(1 - x)^2$. This is more consistent with the view that the distance is a measure of disutility, rather than a representation of geographical distance. Under the latter interpretation, it seems more reasonable to assume that the consumer could go first to the firm located at 0 and then go to the one located at 1 making the total transportation cost equal to $tx^2 + t$ (although one can think of situations where this is not true). Previous models did not have to make this distinction, since consumers in those models bought from one firm exclusively.

Firm prices are denoted by p_A and p_B. We assume that the market is covered, i.e., $V(1)$ is sufficiently high so that each consumer buys at least one brand. If a consumer who is located at x buys from firm A, his indirect utility is $V(1) - tx^2 - p_A$; if he buys from firm B, his indirect utility is $V(1) - t(1 - x)^2 - p_B$; and if he buys from both firms, his indirect utility is $V(2) - [tx^2 + t(1 - x)^2] - p_A - p_B$. There will be two marginal consumers, one denoted by x_1, who is indifferent between buying from firm A only and buying from both firms and the other, denoted by x_2, who is indifferent between buying from firm B exclusively and buying from both firms (see Figure 1). The first marginal consumer is located at,

$$(1) \qquad V(1) - tx_1^2 - p_A = V(2) - tx_1^2 - t(1 - x_1)^2 - p_A - p_B \Leftrightarrow x_1 = 1 - \sqrt{\frac{\theta - p_B}{t}}.$$

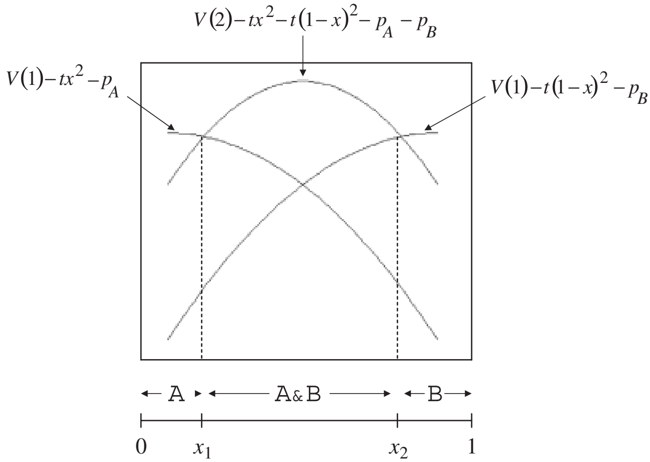

Figure 1
Indirect utilities

The second marginal consumer is located at,

$$V(1) - t(1-x_2)^2 - p_B = V(2) - tx_2^2 - t(1-x_2)^2 - p_A - p_B \tag{2}$$

$$\Leftrightarrow x_2 = \sqrt{\frac{\theta - p_A}{t}}.$$

Note that,

$$x_2 \geqslant x_1 \Leftrightarrow p_A \leqslant p_B - t + 2\sqrt{t(\theta - p_B)}. \tag{3}$$

Also,

$$x_1 \geqslant 0 \Leftrightarrow p_B \geqslant \theta - t, \tag{4}$$

and

$$x_2 \leqslant 1 \Leftrightarrow p_A \geqslant \theta - t, \tag{5}$$

When $x_2 = x_1$, then the marginal consumer is indifferent between buying one unit from firm A and one unit from firm B. In this case, no consumer purchases from both firms. Therefore, the marginal consumer is located at,

$$\hat{x} = x_1 = x_2 = \frac{p_B - p_A + t}{2t}.$$

Hence, the consumers who are located in $[0, x_1]$ purchase firm A's product exclusively, the consumers in (x_1, x_2) purchase from both firms and the ones in $[x_2, 1]$ buy only firm B's product (see Figure 1).

It then follows that firm A's demand function is,

$$d_A = \begin{cases} x_2 = \sqrt{\frac{\theta - p_A}{t}}, & \textit{if } p_A \leqslant p_B - t + 2\sqrt{t(\theta - p_B)}, \\ \hat{x} = \frac{p_B - p_A + t}{2t}, & \textit{if } p_A \geqslant p_B - t + 2\sqrt{t(\theta - p_B)}. \end{cases}$$

Firm B's demand function is,

$$d_B = \begin{cases} 1 - x_1 = \sqrt{\frac{\theta - p_B}{t}}, & \text{if } p_A \leqslant p_B - t + 2\sqrt{t(\theta - p_B)}, \\ 1 - \hat{x} = \frac{p_A - p_B + t}{2t}, & \text{if } p_A \geqslant p_B - t + 2\sqrt{t(\theta - p_B)}. \end{cases}$$

Also, $d_i \in [0, 1]$, $i = A,B$, a condition we have ignored so far. Until the proof of proposition 1, we implicitly assume that $d_i \in (0, 1)$. We deal with the corner solutions in the proof of that proposition.

Note that the demand functions exhibit a kink at $p_A = p_B - t + 2\sqrt{t(\theta - p_B)}$, i.e., they are non-differentiable. For low prices, each firm is a local monopolist in the sense that a firm's demand depends only on its own price. In this case, a reduction in price by one firm does not hurt the demand of the rival. Consumers do not switch brands, but simply more consumers find it advantageous to buy both brands (*aggregate demand creation* effect). As prices increase and after the kink of the demand function, firms compete head-on for consumers and a reduction in price induces some consumers to switch brands (*business stealing* effect).

We assume that firms have constant and equal marginal costs, which are normalized to zero. Thus, the profit functions are given by,

$$(6) \qquad \pi_A = \begin{cases} \sqrt{\frac{\theta - p_A}{t}} p_A, & \text{if } p_A \leqslant p_B - t + 2\sqrt{t(\theta - p_B)}, \\ \frac{(p_B - p_A + t)p_A}{2t}, & \text{if } p_A \geqslant p_B - t + 2\sqrt{t(\theta - p_B)}, \end{cases}$$

and

$$(7) \qquad \pi_B = \begin{cases} \sqrt{\frac{\theta - p_B}{t}} p_B, & \text{if } p_A \leqslant p_B - t + 2\sqrt{t(\theta - p_B)}, \\ \frac{(p_A - p_B + t)p_B}{2t}, & \text{if } p_A \geqslant p_B - t + 2\sqrt{t(\theta - p_B)}. \end{cases}$$

III. ANALYSIS

First, note that the profit functions are not quasi-concave in the strategic variable. Figure 2 depicts the two functions, as given by (6), when $p_B = 1/2$, $\theta = .9$ and $t = 1$.

The profit function is the upper envelope of these two functions and it is clearly not quasi-concave. Therefore, the assumptions of Kakutani's fixed point theorem are not satisfied. In particular, the best replies may not be

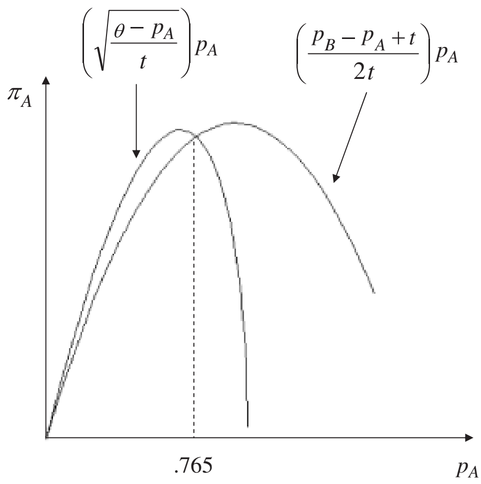

Figure 2
Firm A's profit function

convex-valued. Nevertheless, the game is supermodular[7] and therefore the best replies are increasing. Hence, an equilibrium in pure strategies must exist (see Vives [1999, Theorem 2.5, p. 33]). The analysis which ensues verifies this and furthermore characterizes the equilibrium of the game completely.

We differentiate (6) and (7) with respect to p_A and p_B respectively, set the derivative equal to zero and solve with respect to each firm's strategic variable. This yields,

$$p_A = \begin{cases} \frac{2\theta}{3}, & \text{if } p_A \leqslant p_B - t + 2\sqrt{t(\theta - p_B)} \\ \frac{p_B + t}{2}, & \text{if } p_A \geqslant p_B - t + 2\sqrt{t(\theta - p_B)}, \end{cases}$$

and

$$p_B = \begin{cases} \frac{2\theta}{3}, & \text{if } p_A \leqslant p_B - t + 2\sqrt{t(\theta - p_B)} \\ \frac{p_A + t}{2}, & \text{if } p_A \geqslant p_B - t + 2\sqrt{t(\theta - p_B)}. \end{cases}$$

Let's look at firm A. Firm B's problem will be symmetric. Fix p_B. Firm A has two choices: i) to set $p_A = \frac{2\theta}{3}$, provided that $p_A \leqslant p_B - t + 2\sqrt{t(\theta - p_B)}$, or ii) to set $p_A = \frac{p_B + t}{2}$, provided that $p_A \geqslant p_B - t + 2\sqrt{t(\theta - p_B)}$. The first

[7] See Vives [1999, 2.2.3] for a definition of a supermodular game. It can be easily checked that our game satisfies the conditions of a supermodular game.

choice yields profits equal to $\sqrt{\frac{4\theta^3}{27t}}$ and is valid for,

$$x_2 \geqslant x_1 \Leftrightarrow \frac{2\theta}{3} \leqslant p_B - t + 2\sqrt{t(\theta - p_B)} \Leftrightarrow p_B \leqslant \frac{2(\theta + \sqrt{3\theta t})}{3} - t.$$

The second choice yields profits equal to $\frac{(p_B+t)^2}{8t}$ and is valid for,

$$x_1 = x_2 \Leftrightarrow \frac{p_B + t}{2} \geqslant p_B - t + 2\sqrt{t(\theta - p_B)}$$
$$\Leftrightarrow p_B \geqslant 4\sqrt{t(t+\theta)} - 5t.$$

Hence for $p_B \in \left[4\sqrt{t(t+\theta)} - 5t, \frac{2(\theta+\sqrt{3\theta t})}{3} - t\right]$ both choices satisfy the requirements. In this case, the best response is the one which yields the higher profits. It can be shown that the first choice yields higher profits when $p_B \leqslant \frac{4\sqrt[4]{3t\theta^3}}{3} - t$, while when $p_B \geqslant \frac{4\sqrt[4]{3t\theta^3}}{3} - t$, the second choice yields higher profits. Therefore, the best reply correspondences are given by,

$$(8) \qquad p_A = \begin{cases} \frac{2\theta}{3}, & \text{if } p_B \leqslant \frac{4\sqrt[4]{3t\theta^3}}{3} - t, \\ \frac{p_B+t}{2}, & \text{if } p_B \geqslant \frac{4\sqrt[4]{3t\theta^3}}{3} - t. \end{cases}$$

and

$$(9) \qquad p_B = \begin{cases} \frac{2\theta}{3}, & \text{if } p_A \leqslant \frac{4\sqrt[4]{3t\theta^3}}{3} - t, \\ \frac{p_A+t}{2}, & \text{if } p_A \geqslant \frac{4\sqrt[4]{3t\theta^3}}{3} - t. \end{cases}$$

Figure 3 presents firm A's best reply correspondence as it is given by (8). For p_B's less than $\frac{4\sqrt[4]{3t\theta^3}}{3} - t$, firm A's best response is to charge $p_A = \frac{2\theta}{3}$. This is the region where firm A has a local monopoly and some consumers purchase both brands. At $p_B = \frac{4\sqrt[4]{3t\theta^3}}{3} - t$, firm A has two equally profitable strategies: i) to offer its product at a low price, $\frac{2\theta}{3}$, and sell even to those consumers whose preferences for its brand are not so strong, or ii) to increase the price to $\frac{2\sqrt[4]{3t\theta^3}}{3}$, and focus on the more loyal group of consumers. For any $p_B > \frac{4\sqrt[4]{3t\theta^3}}{3} - t$, firm A's best response is $p_A = \frac{p_B+t}{2}$. This is the region where firms compete head-on for consumers and no consumer buys from both firms. The best reply correspondence indeed never jumps down (provided that $\theta \leqslant 3t$). If $\theta > 3t$, then we reach a corner solution (see proposition 1).

Firm B's best reply correspondence can be obtained in an analogous manner. In part 1 in the statement of proposition 1 below, there are two Nash equilibria which can be Pareto ranked. We assume that the firms can coordinate their play on the better equilibrium, which is the one we report

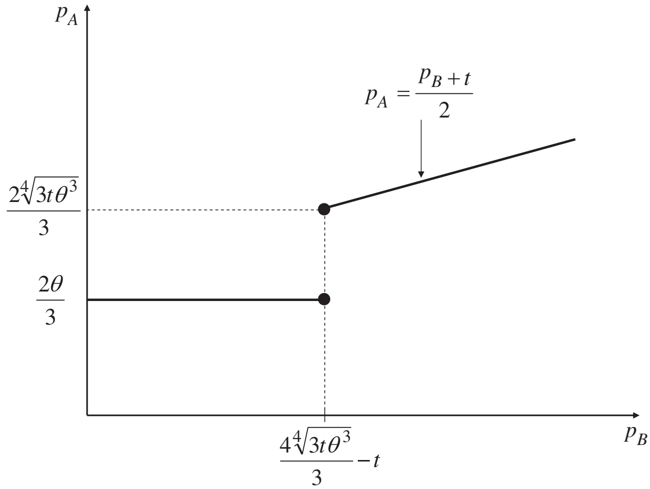

Figure 3
Firm A's best reply correspondence

(see the proof of proposition 1 where we derive both equilibria). The next proposition summarizes the equilibrium.

Proposition 1. The Nash equilibrium prices and profits are as follows:

1. *If* $0 \leqslant \theta \leqslant \frac{6^{2/3}3^{1/3}t}{4} \approx 1.191t$, *then the equilibrium is,*

$$(p_A,\ p_B) = (t,\ t) \text{ and } (\pi_A,\ \pi_B) = \left(\frac{t}{2},\ \frac{t}{2}\right).$$

2. *If* $\frac{6^{2/3}3^{1/3}t}{4} \approx 1.191t < \theta \leqslant 3t$, *then the unique equilibrium is,*

$$(p_A,\ p_B) = \left(\frac{2\theta}{3},\ \frac{2\theta}{3}\right) \text{ and } (\pi_A,\ \pi_B) = \left(\frac{2\sqrt{3}\theta^{\frac{3}{2}}}{9\sqrt{t}},\ \frac{2\sqrt{3}\theta^{\frac{3}{2}}}{9\sqrt{t}}\right).$$

3. *If* $\theta > 3t$, *then the unique equilibrium is,*

$$(p_A,\ p_B) = (\theta - t,\ \theta - t) \text{ and } (\pi_A,\ \pi_B) = (\theta - t,\ \theta - t).$$

Proof. The proof will be based on the best reply correspondences. There are three critical points that decide the relative position of the two best reply correspondences and consequently the equilibrium (see Figure 3): $\frac{2\sqrt[4]{3t\theta^3}}{3}$, $\frac{2\theta}{3}$

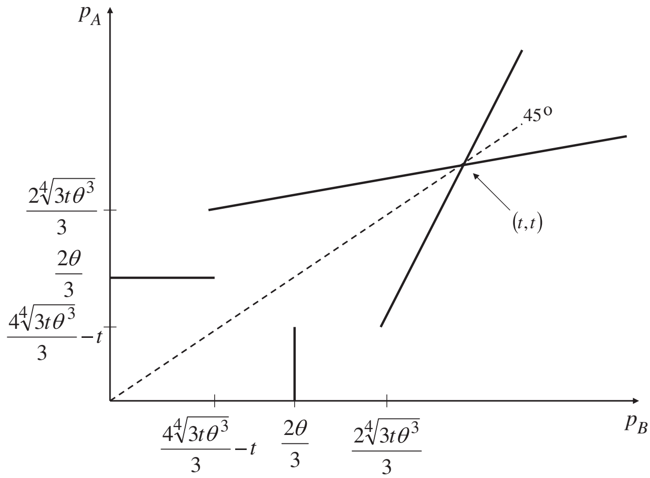

Figure 4
Case 1

and $\frac{4\sqrt[4]{3t\theta^3}}{3} - t$. Recall that $\frac{2\sqrt[4]{3t\theta^3}}{3} \geqslant \frac{2\theta}{3}$. What we have to determine is where $\frac{4\sqrt[4]{3t\theta^3}}{3} - t$ is located relative to these two points. To this end, we have to examine the following four cases.

- *Case 1:* $0 \leqslant \theta < thresh_1 \approx .867t$.[8]

It can be easily calculated that,

$$\frac{2\theta}{3} > \frac{4\sqrt[4]{3t\theta^3}}{3} - t \Leftrightarrow \theta < thresh_1 \approx .867t.$$

Figure 4 represents the equilibrium. The unique Nash equilibrium prices and profits are,

$$(10) \qquad (p_A,\ p_B) = (t,\ t) \text{ and } (\pi_A,\ \pi_B) = \left(\frac{t}{2},\ \frac{t}{2}\right).$$

This is the standard Hotelling equilibrium. For this equilibrium to be valid, it must be that $x_1 = x_2$, or from (3) we must have $p_A \geqslant p_B - t + 2\sqrt{t(\theta - p_B)}$. This holds since, $t \geqslant 2\sqrt{t(\theta - t)} \Leftrightarrow \theta \leqslant \frac{5t}{4}$ a

[8] Whenever, in the remaining of the paper, we present *only* an approximate solution for a threshold (denoted by $\approx$) it is the case that we were not able to obtain an analytic solution. To derive the approximate solution, we used the *fsolve* command in Maple. The thresholds in all of the cases considered in this paper are unique and the approximate solution is arbitrarily close to the true solution.

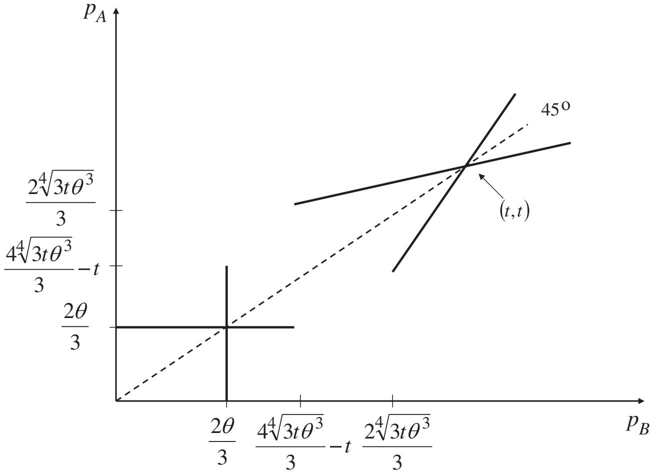

Figure 5
Case 2

condition that is satisfied in this case. Therefore, no consumer buys from both firms and each firm's market share is 1/2.

- *Case 2: $thresh_1 \approx .867t \leqslant \theta \leqslant thresh_2 = \frac{6^{2/3}3^{1/3}t}{4} \approx 1.191t$.*

It can easily be calculated that,

$$\frac{2\sqrt[4]{3t\theta^3}}{3} \geqslant \frac{4\sqrt[4]{3t\theta^3}}{3} - t \Leftrightarrow \theta \leqslant thresh_2 = \frac{6^{2/3}3^{1/3}t}{4} \approx 1.191t.$$

Figure 5 represents the equilibria. There are two Nash equilibria,

$$(p_A,\ p_B) = (t,\ t) \text{ and } (p_A,\ p_B) = \left(\frac{2\theta}{3},\ \frac{2\theta}{3}\right).$$

The associated equilibrium profits are,

$$(\pi_A,\ \pi_B) = \left(\frac{t}{2},\ \frac{t}{2}\right) \text{ and } (\pi_A,\ \pi_B) = \left(\frac{2\sqrt{3}\theta^{\frac{3}{2}}}{9\sqrt{t}},\ \frac{2\sqrt{3}\theta^{\frac{3}{2}}}{9\sqrt{t}}\right). \tag{11}$$

It can easily be calculated that the first equilibrium yields higher profits provided that $\theta \leqslant \frac{6^{2/3}3^{1/3}t}{4}$, a condition that is satisfied in this case.

The first equilibrium, following the same logic as in case 1, is valid. For the second one to be valid, it must be that $x_2 \geqslant x_1$. Given the equilibrium prices,

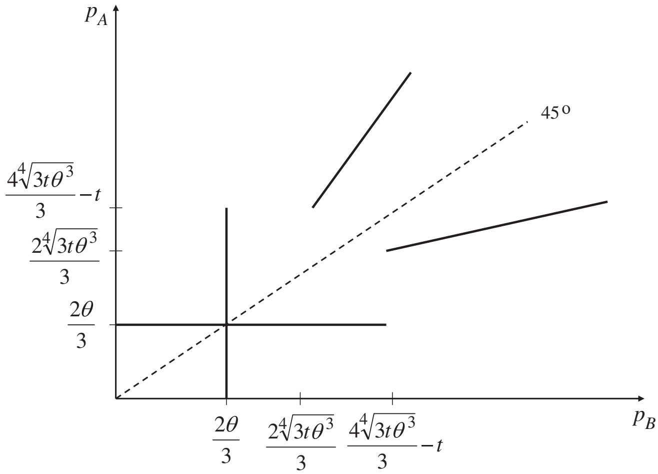

Figure 6
Case 3

and using (1) and (2), we obtain the equilibrium cutoffs,

$$(12,13)\quad x_1^* = 1 - \frac{\sqrt{\theta}}{\sqrt{3t}} \text{ and } x_2^* = \frac{\sqrt{\theta}}{\sqrt{3t}}.$$

Note that $x_2^* > x_1^*$ provided that $\theta > \frac{3t}{4}$ a condition which is satisfied. Hence, the consumers in $\left(x_1^*,\ x_2^*\right)$ buy from both firms. Moreover, if $\theta < 3t$, then $x_1^* > 0$ and $x_2^* < 1$ (the same applies to case 3 below).

- *Case 3: thresh*$_2 = \frac{6^{2/3}3^{1/3}t}{4} \approx 1.191t < \theta \leqslant$ *thresh*$_3 = 3t$.

Figure 6 represents the equilibrium. The unique Nash equilibrium prices and profits are,

$$(p_A, p_B) = \left(\frac{2\theta}{3}, \frac{2\theta}{3}\right) \text{ and } (\pi_A, \pi_B) = \left(\frac{2\sqrt{3}\theta^{\frac{3}{2}}}{9\sqrt{t}}, \frac{2\sqrt{3}\theta^{\frac{3}{2}}}{9\sqrt{t}}\right).$$

- *Case 4*: $\theta >$ thresh$_3$ = 3t.

This is the boundary case. All consumers buy both products. Firms set their prices such that $x_2 = 1$ and $x_1 = 0$. Using (1) and (2), this yields,

$$(p_A,\ p_B) = (\theta - t,\ \theta - t).$$

Since all consumers buy both products, the equilibrium profits are the same as the prices,

$$(\pi_A,\ \pi_B) = (\theta - t,\ \theta - t).$$

To sum up, when $thresh_1 \approx .867t \leqslant \theta \leqslant thresh_2 = \frac{6^{2/3}3^{1/3}t}{4} \approx 1.191t$, there are two Nash equilibria in pure strategies, $(p_A, p_B) = (t, t)$ and $(p_A, p_B) = \left(\frac{2\theta}{3}, \frac{2\theta}{3}\right)$. Moreover, the first one is the Pareto better equilibrium and consequently it will be the one preferred by the firms. This is a property of supermodular games, when the payoff to a player is increasing in the strategy of the other player, as it is the case in our model [see Vives, Remark 14, p. 34]. We assume that firms are able to coordinate their play on the better equilibrium.

Therefore, the non-cooperative outcome is described as follows:

1. If $0 \leqslant \theta \leqslant \frac{6^{2/3}3^{1/3}t}{4}$, then $(p_A, p_B) = (t, t)$, $(\pi_A, \pi_B) = (t/2, t/2)$ and $x_1^* = x_2^* = \frac{1}{2}$ [*no consumer purchases both brands*].
2. If $\frac{6^{2/3}3^{1/3}t}{4} < \theta \leqslant 3t$, then $(p_A, p_B) = \left(\frac{2\theta}{3}, \frac{2\theta}{3}\right)$, $(\pi_A, \pi_B) = \left(\frac{2\sqrt{3}\theta^{\frac{3}{2}}}{9\sqrt{t}}, \frac{2\sqrt{3}\theta^{\frac{3}{2}}}{9\sqrt{t}}\right)$ and [from (12) and (13)] $0 \leqslant x_1^* = 1 - \frac{\sqrt{\theta}}{\sqrt{3t}} < x_2^* = \frac{\sqrt{\theta}}{\sqrt{3t}} \leqslant 1$ [*some consumers purchase both brands*].
3. If $\theta > 3t$, then $(p_A, p_B) = (\theta - t, \theta - t)$, $(\pi_A, \pi_B) = (\theta - t, \theta - t)$ and $0 = x_1^* < x_2^* = 1$ [*all consumers consume both brands*]. ■

There are two types of equilibria. The first is the standard Hotelling outcome where no consumer purchases from both firms. In the second type of equilibrium, some consumers purchase both brands. Each type of equilibrium behaves differently when products become more differentiated, i.e., as t increases. Profits associated with the first type of equilibrium increase, while, somewhat surprisingly, profits associated with the second type decrease. Moreover, in case 4 prices also decrease as products become more differentiated. In the second type of equilibrium, firms benefit when products move closer to each other. Demand increases since consumers find it less costly to consume one more product and for this to happen, a firm does not have to lower its price. The consumers in (x_1, x_2) act as a buffer which lessens, up to a certain extent, the intensity of price competition. When a firm lowers its prices, for example, the demand for its product increases but not at the expense of its rival. Simply, more consumers find it beneficial to incur the incremental transportation cost and consume both products instead of one. This changes the nature of price competition.[9]

[9] According to an analyst, after Apple introduced its first low-priced Macintosh in January of 2005, 'Apple's consumers were probably not going to give up their Windows PC's but might buy a Macintosh as an additional computer for entertainment. It's not about switching but adding. People may still need a PC because of work activities, but this is for doing multimedia activities and searching the Internet.' 'Apple Changes Course with Low-Priced Mac,' *The New York Times*, January 12, 2005.

Welfare analysis

The socially optimal outcome can be found as follows. Since a price is only a transfer and marginal cost is assumed to be zero, a social planner will choose the locations of the two marginal consumers to maximize the utility minus the transportation cost,

$$\max_{(x_1,x_2)} \int_0^{x_1} \left[V(1) - tx^2\right] dx + \int_{x_1}^{x_2} \left[V(2) - tx^2 - t(1-x)^2\right] dx + \int_{x_2}^{1} \left[V(1) - t(1-x)^2\right] dx$$

$$(14) \qquad = \max_{(x_1,x_2)} \frac{tx_1^3}{3} - \frac{tx_2^2}{3} - tx_1^2 + \theta x_2 - \theta x_1 + tx_1 - \frac{t}{3} + V(1).$$

The locations which maximize social welfare are,

$$(15) \qquad x_1^{so} = 1 - \frac{\sqrt{\theta}}{\sqrt{t}} \text{ and } x_2^{so} = \frac{\sqrt{\theta}}{\sqrt{t}}.$$

Note that $x_2^{so} > x_1^{so}$ if and only if $\theta > 1/4$, otherwise it is socially optimal that no consumer purchases both brands, i.e., $x_1 = x_2$. Next, we compare the non-cooperative outcome with the social optimum.

If $0 \leqslant \theta \leqslant t/4$, the the non-cooperative outcome is efficient. It is socially optimal that no consumer purchases both brands, which coincides with the non-cooperative outcome. If $\frac{t}{4} < \theta \leqslant \frac{6^{2/3}3^{1/3}t}{4} \approx 1.191t$, then the non-cooperative outcome is inefficient. In the socially optimal outcome some consumers buy from both firms, but in the non-cooperative equilibrium, each consumer still buys exclusively from one firm. Output is below its efficient level. If $\frac{6^{2/3}3^{1/3}t}{4} \approx 1.191t < \theta < 3t$, then the non-cooperative outcome is inefficient. Although, in the non-cooperative outcome, some consumers consume both products now, they are fewer relative to the number that is socially desired, since $x_2^{so} > x_2^*$ and $x_1^{so} < x_1^*$. Again, output is below its efficient level. Finally, if $\theta \geqslant 3t$, then the non-cooperative outcome is efficient. Efficiency dictates that all consumers should buy from both firms, a situation which is supported when firms behave non-cooperatively.

We see that for intermediate values of θ, there is a deadweight loss associated with the Nash equilibrium, unlike the standard Hotelling model which yields an efficient outcome (if the firm locations are fixed and the market is covered).

IV. PRODUCT SELECTION

We consider a two-stage game. In stage 1, firms choose their locations on the unit interval. Let a (the location of firm A) and b (the location of firm B)

denote the distance from 0. We assume that $1 \geqslant b \geqslant \frac{1}{2} \geqslant a \geqslant 0$.[10] In stage 2, they choose prices. The next proposition summarizes the subgame perfect equilibrium (SPE).[11] For simplicity, and without much loss of generality, we set $t = 1$. To make the analysis easier and more transparent, we first assume (in proposition 2) that θ is independent of a and b. After proposition 2, we also discuss how our results are affected if we allow θ to depend on the firms' locations.

Proposition 2.

If $0 \leqslant \theta \leqslant thresh \approx .711$ then the SPE is described as follows,[12]

- *Stage 1:* $a = 0$ *and* $b = 1$ *(maximal differentiation).*
- *Stage 2:* $p_A = p_B = 1$ *and* $\pi_A = \pi_B = 1/2$.

If $5/4 \geqslant \theta > thresh \approx .711$ then the symmetric SPE is described as follows,[13]

- *Stage 1:* $a = b = 1/2$ *(minimal differentiation).*
- *Stage 2:* $p_A = p_B = \frac{2\theta}{3} + \frac{\sqrt{1+12\theta}-1}{18}$ *and* $\pi_A = \pi_B = \frac{\left(2+\sqrt{1+12\theta}\right)\left(12\theta-1+\sqrt{1+12\theta}\right)}{108}$

Proof. See appendix. ■

If θ is independent of a and b, and higher than $thresh \approx .711$, then firms will have the tendency to move next to each other, i.e., $a = b = 1/2$. The intuition is as follows. When a firm is moving towards the middle, it increases its demand by making its product more attractive to those who prefer the rival's brand. This, however, does not lead to an all-out competition, since the increase in demand does not imply lower sales for the rival firm (*aggregate*

[10] The assumption that $b \geqslant 1/2$ and $a \leqslant 1/2$ is made for simplicity. Without it, the analysis becomes unnecessarily more complicated. Moreover, with this assumption we can rule out asymmetric minimal differentiation equilibria and focus on the more important results and intuition.

[11] If a consumer who is located at x, with $x < a < b$, buys from both firms, the disutility he incurs is $(a - x)^2 + (b - x)^2$. As we argued in section II, this is more consistent with the view that the distance is a measure of disutility, rather than a representation of geographical distance.

[12] An analytic expression for this threshold exists and it is given by,

$$-\frac{1}{12} + \frac{1}{12}\left[\left(28 + 3\sqrt{87}\right)^{1/3} + \frac{1}{\left(28 + 3\sqrt{87}\right)^{1/3}} - 1\right]^2.$$

[13] If $\theta \geqslant 5/4$, then we reach a corner solution where all consumers purchase both brands, i.e., $x_1 = 0$ and $x_2 = 1$. Minimal differentiation continues to hold in this case. The corner solution, however, is not very interesting and therefore it is omitted.

demand creation). As a consequence, the rival has no incentive to lower its price since its customers are not switching brands but simply are buying both, and in a way it accommodates the firm's movement towards the center.

From (A6), the marginal consumers (when $a = b = 1/2$) are located at,

$$x_1^{**} = \frac{2}{3} - \frac{\sqrt{1+12\theta}}{6} \text{ and } x_2^{**} = \frac{1}{3} + \frac{\sqrt{1+12\theta}}{6}. \tag{16}$$

Note that if $\theta < \frac{5}{4}$, then $x_1^{**} > 0$ and $x_2^{**} < 1$. If $\theta \geqslant \frac{5}{4}$, then $x_1^{**} = 0$ and $x_2^{**} = 1$, i.e., all consumers purchase both brands. The socially optimal locations are given by (15). As we saw in Section 3, when firms are located at the extremes, $x_1^{so} \leqslant x_1^*$ and $x_2^{so} \geqslant x_2^*$ and the non-cooperative outcome is (weakly) inefficient. Moreover, $x_1^{so} \leqslant x_1^{**} \leqslant x_1^*$ and $x_2^{so} \geqslant x_2^{**} \geqslant x_2^*$. Since the welfare function [see (14)] is concave in x_1 and x_2, efficiency improves when firms are free to choose their positions on the unit interval (i.e., more output is produced).

The incremental utility θ is a function of the firms' locations.

So far, for simplicity and clarity, we have assumed that the incremental utility, θ, is independent of the spatial locations of the two firms and we showed that if θ exceeds *thresh* $\approx .711$, then the spatial differentiation will be eliminated, while if $\theta \leqslant$ *thresh* $\approx .711$, differentiation will be maximal (as in d'Aspermont *et al.*). Instead, we can assume that θ depends on a and b, i.e., $\theta(a, b)$. One reasonable assumption is that θ is decreasing in a and increasing in b (i.e., as products move closer to each other, the incremental utility declines). The equilibrium profit functions (if some consumers make multiple purchases), from (A4) and (A5), can be re-written as follows [where θ is replaced by $\theta(a, b)$],[14]

$$\pi_A = \frac{2\left(2a + \sqrt{a^2 + 3\theta(a, b)}\right)\left(3\theta(a, b) - a^2 + a\sqrt{a^2 + 3\theta(a, b)}\right)}{27} \text{ and}$$

$$\pi_B = \frac{2\left(2(1-b) + \sqrt{(1-b)^2 + 3\theta(a, b)}\right)\left(3\theta(a, b) - (1-b)^2 + (1-b)\sqrt{(1-b)^2 + 3\theta(a, b)}\right)}{27}.$$

This creates an interesting tension as far as the locations are concerned. From the proof of proposition 2, we know that as firms move closer to each other, their profits increase if θ is kept fixed, i.e., $\frac{\partial \pi_A}{\partial a}\big|_{\theta=\bar{\theta}} > 0$ and $\frac{\partial \pi_B}{\partial b}\big|_{\theta=\bar{\theta}} < 0$, *positive effect*. But, on the other hand, as firms move closer to each other, θ

[14] If consumers do not make multiple purchases in equilibrium, then the profit functions do not depend on θ, see (A8).

decreases, $\frac{\partial\theta}{\partial a}<0$ and $\frac{\partial\theta}{\partial b}>0$. Moreover, from the proof of proposition 2, we also know that, $\frac{\partial\pi_A}{\partial\theta}>0$ and $\frac{\partial\pi_B}{\partial\theta}>0$ [see (A4) and (A5)]. Hence, we have identified a second *negative effect* on the profits as firms move closer to each other, $\frac{\partial\pi_A\partial\theta}{\partial\theta\,\partial a}<0$ and $\frac{\partial\pi_B\partial\theta}{\partial\theta\,\partial b}>0$. Consequently, the total effect can be expressed as follows,

$$(17)\qquad \frac{\partial\pi_A(a,b)}{\partial a}=\underbrace{\left.\frac{\partial\pi_A}{\partial a}\right|_{\theta=\bar{\theta}}}_{(+)}+\underbrace{\frac{\partial\pi_A}{\partial\theta}\frac{\partial\theta}{\partial a}}_{(-)} \text{ and}$$

$$\frac{\partial\pi_B(a,b)}{\partial b}=\underbrace{\left.\frac{\partial\pi_B}{\partial b}\right|_{\theta=\bar{\theta}}}_{(-)}+\underbrace{\frac{\partial\pi_A}{\partial\theta}\frac{\partial\theta}{\partial b}}_{(+)}.$$

The overall effect of the firms' locations on profits will depend on the relative strength of these two effects, which in turn depends on the functional form of $\theta(a, b)$.

We attempt to shed more light onto the problem with the aid of the following two examples. First, consider the following simple functional form for θ,

$$\theta(a,b)=\begin{cases}1.5, & \text{if } b-a\geqslant k\\ 0, & \text{if } b-a<k,\end{cases}$$

where $0 < k\leqslant 1$. This functional form allows for the possibility that when the firms locate very close to each other, the incremental utility from consuming a second product goes to zero. Following proposition 2, it can readily be verified that in equilibrium $b-a=k$. Therefore, spatial differentiation will be neither maximal nor minimal. Firms benefit by moving close to each other to induce some consumers to make multiple purchases, but not too close because then products are too similar and consumer's incremental utility from consuming a second product falls drastically.

As a second example, suppose that $\theta(a, b) = 1.2(b-a)^{.1}$. We substitute this function into π_A and π_B and we show (numerically) that in the symmetric SPE: i) locations $a\approx .445$ and $b\approx .555$. and ii) prices, $p_A=p_B\approx .77$. Equilibrium profits are, $\pi_A=\pi_B\approx .68$. We see again that spatial differentiation is neither maximal nor minimal (*intermediate*).

We would like to emphasize at this point that $a=b$ *does not necessarily* imply $\theta=0$, as we assumed in the previous two examples. For instance, products may be differentiated along other (non-spatial) dimensions, or simply a second product may yield a positive incremental utility, even in the absence of horizontal differentiation. For example, even if *all* consumers are willing to pay the same for two credit cards (no horizontal differentiation), subscribing to both cards provides consumers a positive incremental utility (due to higher credit limit). Another example is DVD movies which are differentiated with respect to how they mix (say) violence with romance. The mix could be identical (no horizontal differentiation) but consumers may still enjoy a positive incremental utility from watching both movies. Thus, there are markets where consumers derive positive utility from consuming a second product that is in many respects similar to the first one. Nevertheless, for a number of products and markets, $a = b$ will imply $\theta = 0$, which will further imply that firms will not locate next to each other. Spatial differentiation in this case may be intermediate. After all, the specific properties of the function $\theta(a, b)$ will depend on a given market and they are a matter of empirical research. A novel result of this paper is that when we allow consumers to purchase from both firms, in a Hotelling-type model, horizontal differentiation may not be maximal.

Next, we summarize our findings regarding the firms' locations on the interval.

- Suppose θ is independent of a and b. Then, if $\theta \leqslant thresh \approx .711$, differentiation will be maximal, while if $\theta > thresh \approx .711$, differentiation will be minimal. The latter result stands in contrast with the standard horizontal differentiation model with quadratic transportation costs where differentiation is maximal.

Now suppose θ does depend on a and b and in particular $\frac{\partial\theta}{\partial a} < 0$ and $\frac{\partial\theta}{\partial b} > 0$. Then we can say the following:

- If $\theta(0,1) \geqslant \frac{6^{2/3}3^{1/3}t}{4} \approx 1.191$, $\left|\frac{\partial\theta(0,1)}{\partial a}\right| < \frac{2\sqrt{\theta(0,1)}}{\sqrt{3}}$ and/or $\left|\frac{\partial\theta(0,1)}{\partial b}\right| < \frac{2\sqrt{\theta(0,1)}}{\sqrt{3}}$, then it is guaranteed that differentiation will *not* be maximal. This can be seen as follows. From the proof of proposition 1, we know that if $\theta = \frac{6^{2/3}3^{1/3}t}{4}$, then firms' equilibrium profits are the same whether some consumers make multiple purchases or not, provided that $a = 0$ and $b = 1$. Moreover, from (17) we have $\frac{\partial\pi_A(0,1)}{\partial a} = \frac{2\theta(0,1)}{3} + \frac{\sqrt{\theta(0,1)}}{\sqrt{3}}\frac{\partial\theta(0,1)}{\partial a} > 0 \Rightarrow \left|\frac{\partial\theta(0,1)}{\partial a}\right| < \frac{2\sqrt{\theta(0,1)}}{\sqrt{3}}$. A similar condition will hold for firm B. Therefore, if the above conditions are satisfied, profits will increase as firms move closer to each other, in an attempt to create demand in the form of more multiple purchases.

- If $\theta\left(\frac{1}{2},\frac{1}{2}\right) = 0$, then differentiation will *not* be minimal. Profits are zero when both firms are located in the middle.

- If $\theta(0,1) \geqslant \frac{6^{2/3}3^{1/3}t}{4} \approx 1.191$, $\left|\frac{\partial\theta(0,1)}{\partial a}\right| < \frac{2\sqrt{\theta(0,1)}}{\sqrt{3}}$, $\left|\frac{\partial\theta(0,1)}{\partial b}\right| < \frac{2\sqrt{\theta(0,1)}}{\sqrt{3}}$ and $\theta\left(\frac{1}{2},\frac{1}{2}\right) = 0$, then spatial differentiation *will be* intermediate (note that the second example above, where differentiation is intermediate, satisfies these conditions).[15] This can be seen by combining the above two cases.

- Finally, if $\theta\left(\frac{1}{2},\frac{1}{2}\right) > 0$, and (following the logic outlined three paragraphs above)

$$\frac{\partial \pi_A\left(\frac{1}{2},\frac{1}{2}\right)}{\partial a} \geqslant 0 \Leftrightarrow \left|\frac{\partial \theta\left(\frac{1}{2},\frac{1}{2}\right)}{\partial a}\right| \leqslant \frac{\sqrt{1+12\theta\left(\frac{1}{2},\frac{1}{2}\right)}\left[12\theta\left(\frac{1}{2},\frac{1}{2}\right)-1\right]+1+12\theta\left(\frac{1}{2},\frac{1}{2}\right)}{3\left[1+12\theta\left(\frac{1}{2},\frac{1}{2}\right)+2\sqrt{1+12\theta\left(\frac{1}{2},\frac{1}{2}\right)}\right]}$$

[and the same condition for $\frac{\partial \pi_B\left(\frac{1}{2},\frac{1}{2}\right)}{\partial b}$], then differentiation *may* be minimal. As an illustrative example consider, $\theta(a,b) = 1.25 + (b-a)$. In this example the above conditions are satisfied, i.e., $\theta\left(\frac{1}{2},\frac{1}{2}\right) > 0$ and

$$\left|\frac{\partial \theta\left(\frac{1}{2},\frac{1}{2}\right)}{\partial a}\right| = 1 \leqslant \frac{\sqrt{1+12\theta\left(\frac{1}{2},\frac{1}{2}\right)}\left[12\theta\left(\frac{1}{2},\frac{1}{2}\right)-1\right]+1+12\theta\left(\frac{1}{2},\frac{1}{2}\right)}{3\left[1+12\theta\left(\frac{1}{2},\frac{1}{2}\right)+2\sqrt{1+12\theta\left(\frac{1}{2},\frac{1}{2}\right)}\right]}$$
$$= 1.$$

We can show that spatial differentiation is indeed minimal.

V. CONCLUSION

We introduce a duopoly model with differentiated products, where each firm produces one brand. Consumers cannot buy more than one unit from each brand, but they can purchase two brands, one from each firm. This model is capable of generating a very realistic equilibrium where some consumers remain loyal to one brand, while another group of consumers consumes both brands. Products that fit this description include, credit cards, software, subscriptions to magazines, newspapers, scholarly journals and TV channels. First, we ignore the issue of product selection by fixing the firms' locations at the two extremes of the unit interval. There are two types of equilibria: i) the standard Hotelling one, where no consumer buys from both firms and equilibrium profits decrease as products become less differentiated (i.e, as the transportation cost parameter t decreases) and ii) an equilibrium where some consumers purchase from both firms and equilibrium profits increase as products become less differentiated. When the magnitude of the incremental utility from purchasing a second brand is below a threshold, then the

[15] We assume that a location equilibrium (in pure strategies) in stage 1 exists. In general, this depends on the properties of the function $\theta(a,b)$. If $\theta(a,b)$ is constant, then an equilibrium exists (see proposition 2). But for a general $\theta(a,b)$, to derive an existence result, we would need more conditions. We do not pursue the existence issue any further. In the specific examples that we have presented in this section, an equilibrium exists.

equilibrium is of the first type. When the incremental utility exceeds that threshold, then the unique equilibrium is of the second type.

Next, we allow firms to choose their locations on the horizontal dimension. In particular, we analyze a two-stage game where firms position themselves on the interval in stage 1 and in stage 2 they compete in prices. If the incremental utility is below a threshold, then differentiation is maximal. If the incremental utility exceeds that threshold, then the equilibrium is characterized by minimum differentiation. Hence, this model can restore Hotelling's *Principle of Minimum Differentiation*. Our model has identified an *aggregate demand creation* effect, which is the driving force behind the firms' tendency to move closer to each other. This aggregate demand creation effect is new and it is not present in single purchase models. Our model may also provide a new explanation as to why retailers, in many cases, choose to locate very close to each other [e.g. big shopping centers and malls]. Finally, if we allow the incremental utility to depend on the location of the two firms, then differentiation may be intermediate.

The model presented in this paper can be extended to shed light on the issue of product variety in a monopolistically competitive setting. When competition is localized, as in Salop's circular model (Salop [1979]), then the number of brands (firms) is excessive from the social point of view. The reason is the lack of global competition, which tends to boost profits and consequently entry. When consumers can buy from more than one firm, as we have postulated in this paper, then firms compete not only with their neighbors, but also with rivals which are not adjacent to them. This retains the spatial differentiation aspect, but injects into the model a realistic dose of non-localized competition. This can potentially change the qualitative features of the equilibrium.

APPENDIX

Proof of proposition 2. Following an argument similar to that in the proof of proposition 1, it can be shown that the best reply correspondences if they have a jump that jump is upward, for any a and b.[16] Therefore, a pure strategy price equilibrium in stage 2 exists, for any a and b. There are two possible outcomes. Either some consumers purchase both products, i.e., $x_2 > x_1$, or no consumer makes multiple purchases, i.e., $x_1 = x_2 = \hat{x}$. We will mainly focus on an interior solution, i.e., $x_1 > 0$ and $x_2 < 1$.

First, suppose that some consumers purchase both products. The marginal consumer x_1 satisfies,

$$V(1) - (a - x_1)^2 - p_A = V(2) - (a - x_1)^2 - (b - x_1)^2 - p_A - p_B \Leftrightarrow x_1 = b \pm \sqrt{\theta - p_B}.$$

To begin with, note that $b \geqslant x_1$. To see this, suppose by way of contradiction that $b < x_1$. Consider consumer x who is located between (x_1, x_2). This consumer prefers

[16] The details are omitted.

buying both products to buying only A, i.e.,

$$V(2) - (a-x)^2 - (b-x)^2 - p_A - p_B > V(1) - (a-x)^2 - p_A. \ (*)$$

Consider now a consumer located at $y \leqslant x_1$. Without any loss of generality set $y = b$. This consumer must prefer brand A, i.e.,

$$V(1) - (a-b)^2 - p_A \geqslant V(2) - (a-b)^2 - p_A - p_B. \ (**)$$

Now note that $V(2) - (a-b)^2 - p_A - p_B > V(2) - (a-x)^2 - (b-x)^2 - p_A - p_B$, since $x > x_1 > b \geqslant a$. By combining (*) and (**) we can conclude that,

$$V(1) - (a-b)^2 - p_A > V(1) - (a-x)^2 - p_A \Rightarrow x < b,$$

a contradiction.

Then, the first marginal consumer, x_1, is given by the following expression,

$$\text{(A1)} \qquad x_1 = b - \sqrt{\theta - p_B}.$$

The second marginal consumer, x_2, satisfies,

$$V(1) - (b-x_2)^2 - p_B = V(2) - (b-x_2)^2 - (a-x_2)^2 - p_A - p_B.$$

By following the same steps as the ones to prove that $b \geqslant x_1$, we can show that $x_2 \geqslant a$. Then,

$$\text{(A2)} \qquad x_2 = a + \sqrt{\theta - p_A}.$$

Firm A's demand function is $d_A = a + \sqrt{\theta - p_A}$ and firm B's is $d_B = 1 - b + \sqrt{\theta - p_B}$. The profit functions are, $\pi_A = p_A d_A$ and $\pi_B = p_B d_B$. Note that the demand functions and therefore the profit functions are strictly concave in prices.

The prices which maximize the profit functions (assuming that some consumers make multiple purchases) are given by,

$$\text{(A3)} \qquad p_A = \theta - \frac{(\sqrt{a^2 + 3\theta} - a)^2}{9} \text{ and } p_B = \theta - \frac{(\sqrt{(1-b)^2 + 3\theta} - 1 + b)^2}{9}.$$

The maximized profits are given by,

$$\text{(A4)} \qquad \pi_A = \frac{2(2a + \sqrt{a^2 + 3\theta})(3\theta - a^2 + a\sqrt{a^2 + 3\theta})}{27} \text{ and}$$

$$\text{(A5)} \qquad \pi_B = \frac{2(2(1-b) + \sqrt{(1-b)^2 + 3\theta})(3\theta - (1-b)^2 + (1-b)\sqrt{(1-b)^2 + 3\theta})}{27}.$$

It follows easily that $\frac{\partial \pi_A}{\partial a} > 0$ and $\frac{\partial \pi_B}{\partial b} < 0$. To see this, observe that d_A and d_B increase as a increases and b decreases respectively. Moreover, each firm is a local monopolist and therefore even if a firm does not adjust its price following an increase in a or a decrease in b, profits will increase. This means that if some consumers purchase both products, then firms will have the tendency to agglomerate in the middle (*minimal differentiation*). Using the same logic, we can also show that $\frac{\partial \pi_A}{\partial \theta} > 0$ and $\frac{\partial \pi_B}{\partial \theta} > 0$.

The marginal consumers, after we substitute (A3) into (A1) and (A2), are given by,

$$\text{(A6)} \qquad x_1 = \frac{1 + 2b - \sqrt{(1-b)^2 + 3\theta}}{3} \text{ and } x_2 = \frac{2a + \sqrt{a^2 + 3\theta}}{3}.$$

Furthermore, $x_1 > 0$ and $x_1 < 1$ if and only if $\theta < b(b+2)$ and $\theta < a(a-4)+3$ respectively.

Second, suppose that no consumer purchases both products. It is well-known (see d'Aspermont *et al.*) that the equilibrium prices and profits in this case are given by,

$$\text{(A7)} \qquad p_A = \frac{(b-a)(2+b+a)}{3} \text{ and } p_B = \frac{(b-a)(4-a-b)}{3}$$

and

$$\text{(A8)} \qquad \pi_A = \frac{(b-a)(2+b+a)^2}{18} \text{ and } \pi_B = \frac{(b-a)(4-a-b)^2}{18}.$$

Profits increase when firms move away from each other (*maximal differentiation*).

It follows from the properties of the profit functions [(A4), (A5) and (A8)] that spatial differentiation will be either maximal or minimal and *nothing* in between. Next, we find conditions under which each product location configuration is a SPE.

Minimal differentiation

Suppose that $a = b = 1/2$. First, it follows from (A6) that $x_1 > 0$ and $x_2 < 1$ if and only if $\theta < 5/4$. The prices, after we substitute $a = b = 1/2$ into (A3), are given by,

$$\text{(A9)} \qquad p_A = p_B = \frac{2\theta}{3} + \frac{\sqrt{1+12\theta} - 1}{18}.$$

The profits, using (A4) and (A5), are given by,

$$\text{(A10)} \qquad \pi_A = \pi_B = \frac{\left(2 + \sqrt{1+12\theta}\right)\left(12\theta - 1 + \sqrt{1+12\theta}\right)}{108}.$$

Next, we examine unilateral deviations in each stage of the game.

Deviation in stage 2

We fix $a = b = 1/2$ and we examine a price deviation. No firm has an incentive to deviate from (A9) in stage 2, if even after a deviation some consumers purchase both brands. The only possible deviation is for a firm to increase its price so that no consumer purchases both brands. But in this case the higher price firm (the one who deviated) will receive zero profits, since $a = b = 1/2$. Hence, no deviation in price is profitable in stage 2.

Deviation in stage 1

The only deviation that a firm will consider is moving to an endpoint of the interval. Let's assume that firm A deviates to $a = 0$. Next, we search for the equilibrium prices in stage 2 (assuming that $a = 0$ and $b = 1/2$).

The equilibrium prices and profits, assuming that $x_2 = x_1$ (i.e., no multiple purchases) and using (A7) and (A8) are given by,

$$p_A = \frac{5}{12} \text{ and } p_B = \frac{7}{12}$$

and

$$\pi_A = \frac{25}{144} \text{ and } \pi_B = \frac{49}{144}.$$

Suppose now that firm A deviates by lowering its price so that some consumers make multiple purchases. The optimal deviation price and profits are given by,

$$p_A^{dev} = \frac{2\theta}{3} \text{ and } \pi_A^{dev} = \frac{2\sqrt{3}\theta^{\frac{3}{2}}}{9}.$$

It turns out that such a deviation is profitable provided that $\theta > \frac{450^{2/3}3^{1/3}}{144} \approx .588$ (which also implies that $x_2 > x_1$). Moreover, $p_A^{dev} = \frac{2\theta}{3} < p_A = \frac{5}{12} \Rightarrow \theta < .625$. Therefore, if $\theta \leqslant \frac{450^{2/3}3^{1/3}}{144} \approx .588$, then firm A has no incentive to deviate.

Next, we look at firm B's incentive to deviate. The optimal price and profits after deviation are given by,

$$p_B^{dev} = \frac{2\theta}{3} + \frac{\sqrt{1+12\theta}-1}{18} \text{ and } \pi_B^{dev} = \frac{\left(2+\sqrt{1+12\theta}\right)\left(12\theta - 1 + \sqrt{1+12\theta}\right)}{108}.$$

It turns out that such a deviation is profitable provided that $\theta \geqslant thresh \approx .5146$ (which also implies that $x_2 > x_1$). Moreover, $p_B^{dev} = \frac{2\theta}{3} + \frac{\sqrt{1+12\theta}-1}{18} < p_B = \frac{7}{12} \Rightarrow \theta < 1 - \frac{\sqrt{51}}{24} \approx .7024$. So, if $\theta \leqslant thresh \approx .5146$, then firm B has no incentive to deviate.

Therefore, if $\theta \leqslant .5146$, then an equilibrium where no consumer makes multiple purchases exists, given that $a = 0$ and $b = 1/2$.

Now assume that some consumers make multiple purchases and a firm deviates by increasing its price so that no consumer makes multiple purchases. The prices and profits before deviation are given by (A3), (A4) and (A5) after we substitute $a = 0$ and $b = 1/2$.

First, we assume that firm A deviates. The optimal deviating price and profits are given by,

$$p_A^{dev} = \frac{\theta}{3} + \frac{7}{72} + \frac{\sqrt{1+12\theta}}{36} \text{ and } \pi_A^{dev} = \frac{\left(24\theta + 7 + 2\sqrt{1+12\theta}\right)^2}{5184}.$$

It turns out that if $\theta \geqslant thresh \approx .325$, then such a deviation is unprofitable. If $\theta \leqslant \frac{15}{4} - \frac{7\sqrt{15}}{8} \approx .361$, then $x_1 = x_2$. Moreover, $p_A^{dev} = \frac{\theta}{3} + \frac{7}{72} + \frac{\sqrt{1+12\theta}}{36} > p_A \Rightarrow \theta < \frac{8+\sqrt{19}}{24} \approx .515$.

Second, we assume that firm B deviates. The optimal deviating price and profits are given by,

$$p_B^{dev} = \frac{\theta}{3} + \frac{3}{8} \text{ and } \pi_B^{dev} = \frac{(9+8\theta)^2}{576}.$$

It turns out that if $\theta \geqslant thresh \approx .4126$, then such a deviation is unprofitable. If $\theta \leqslant \frac{27}{8} - \frac{\sqrt{3}\sqrt{42}}{4} \approx .567$, then $x_1 = x_2$. Moreover, $p_B^{dev} = \frac{\theta}{3} + \frac{3}{4} > p_B \Rightarrow \theta < \frac{35-2\sqrt{70}}{24} \approx .761$.

Therefore, if $\theta \geqslant .4126$, then an equilibrium where some consumers make multiple purchases exists, given that $a = 0$ and $b = 1/2$.

Next, we summarize our findings regarding firm A's deviation in stage 1.

- If $\theta > thresh \approx .5146$, then the unique price equilibrium results in some consumers making multiple purchases. In this case firm A has no incentive to deviate to $a = 0$ in stage 1. This is because even if $a = 0$, still some consumers make multiple purchases in stage 2, and given this profits decrease when a firm moves to the extremes.

- If $\theta < thresh \approx .4126$, then the unique price equilibrium results in no consumer making multiple purchases. Firm A will not deviate in stage 1 if and only if $\pi_A = \frac{\left(2+\sqrt{1+12\theta}\right)\left(12\theta-1+\sqrt{1+12\theta}\right)}{108} \geqslant \pi_A^{dev} = \frac{25}{144}$. If $\theta \geqslant thresh \approx .2877$, then firm A has no incentive to deviate.

- Finally, if $.4126 \approx thresh \leqslant \theta \leqslant thresh \approx .5146$, then there are two equilibria, one where no consumer makes multiple purchases and another where some consumers do buy both products. But, as we have demonstrated above in either case firm A would not want to deviate.

To sum up, $a = b = 1/2$ followed by $p_A = p_B = \frac{2\theta}{3} + \frac{\sqrt{1+12\theta}-1}{18}$, constitutes a SPE in the two-stage game if and only if $\theta \geqslant thresh \approx .2877$.

Maximal differentiation

Suppose now that $a = 0$ and $b = 1$. In this case, the prices and profits before any deviation occurs are given by proposition 1.

Deviation in stage 2

We fix $a = 0$ and $b = 1$ and we examine a price deviation. We know from proposition 1 that an outcome where no consumer makes multiple purchases is an equilibrium if and only if $0 \leqslant \theta \leqslant \frac{6^{2/3}3^{1/3}t}{4} \approx 1.191$.

Deviation in stage 1

Now assume that firm B deviates by moving closer to firm A in order to increase its profits, i.e., $b = 1/2$. The analysis is the same as in the case of minimal differentiation that we examined above (since $a = 0$ and $b = 1/2$). It then follows that firm B will not deviate in stage 1 if and only if $\pi_B = \frac{1}{2} \geqslant \pi_B^{dev} = \frac{\left(2+\sqrt{1+12\theta}\right)\left(12\theta-1+\sqrt{1+12\theta}\right)}{108}$. If $\theta \leqslant thresh \approx .711$, then firm B has no incentive to deviate (even if after its deviation some consumers make multiple purchases).

It follows from the analysis so far that if $.2877 \approx thresh \leqslant \theta \leqslant thresh \approx .711$, then there are two SPE: one characterized by maximal and the other by minimal differentiation. It can be shown that, in this case, the maximal differentiation equilibrium yields higher profits to both firms than the minimal differentiation equilibrium. We will assume that firms are able to coordinate their play on the better equilibrium. Therefore, $a = 0$, $b = 1$ followed by $p_A = p_B = 1$ is the SPE if and only if $\theta \leqslant thresh \approx .711$ and $a = b = 1/1$ followed by $p_A = p_B = \frac{2\theta}{3} + \frac{\sqrt{1+12\theta}-1}{18}$ is the symmetric SPE if and only if $\theta \leqslant thresh \approx .711$. ■

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